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Polarization of Economies in an Endogenous Growth Model of Capital R&D Input

Shiro Kuwahara

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Polarization of Economies in an Endogenous Growth Model of Capital R&D Input*

Shiro Kuwahara[†‡]

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Abstract

This study develops a model in which capital is used in production of final goods and R&D activities. This arrangement can provide new insight into the role of capital in economic growth. The capital stock determines the occurrences of R&D activities, and interest rate subsidies always promote economic welfare and policies guide some countries into poverty traps on a steady-growth path. Development aid through provision of factor stocks such as capital and technology are ineffective for an economy to ride on a steady growth path. Promoting efficient goods production or R&D, on the other hand, is effective.

Keywords: Polarization of economies, Roles of capital, R&D-based growth, Poverty traps, Roles of economic policies.

JEL Classification: E00, O00, O41.

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†Graduate School of Economics, Kyoto University

‡Shiro Kuwahara, E-mail: kuwahala@mbox.kyoto-inet.or.jp, Postal address: Mori Mansion V 305, 21-1 Jizomotocho, Sakyo-ku, Kyoto, 606-8181, JAPAN, Phone Number:+81-75-722-1050, FAX Number:+81-537-23-3024.

1 Introduction

This paper develops a model in which capital is used in production of final goods and R&D activities, and focus on issues of economic development: the polarization of economies into constantly growing and no-growth countries. Wide diversity of economic growth rates has absorbed the attention of many economists (e.g. Lucas 1988). Easterly (1994) and Quah (1996, 1997) extracted polarization of growth rates from this diversity. Polarized economies include advanced countries, which enjoy higher growth rates, and underdeveloped countries, which retain low or zero growth rates. It is said that such countries are mired in poverty traps. Furthermore, the polarization process engenders the decomposition of middle class economies. Some countries grow positively and come to grow steadily, whereas others are caught in poverty traps. The present study is intended to explain these phenomena of polarized and polarizing economies using an R&D-based growth model with capital as an input for R&D production.

When economists began to study growth accounting nearly half a century ago (e.g. Solow 1957), they were surprised with the obtained results: economic growth is mainly attributable to technological progress – capital and labor are not so important. This observation has been incorporated into first generation R&D-based growth models (e.g. Romer 1990) as late as the 1990s. This type of model relates the growth rate and R&D input. Assuming that exogenous human resources are used in R&D activities, first generation R&D-based growth studies often conclude that the introduction of capital would not change most basic results (Grossman and Helpman 1991 Ch5, Aghion and Howitt 1998 Ch3). However, some studies have related capital accumulation to R&D activities. For example, Abramovitz and David (1973) demonstrated that R&D activities positively depends on the level of capital accumulation, and Chandler (1990) demonstrated that the scale expansion of enterprises generates R&D activities and product diversification. Hence, we infer that capital is used in R&D activities,[1] and to demonstrate the role of capital as a determinant of R&D activities.

The model induces several remarkable results regarding the mechanism

[1]It is possible to trace the development of these analyses to Shell (1967, 1973) in the field of neoclassical growth models with capital for R&D. It is nevertheless difficult to assert that sufficient research has been done since that time. The important exception is the lab equipment model of Rivera-Breiz and Romer (1991) and Matsuyama (1999).

the engenders the polarization of economies and effective economic policies for escaping poverty traps. First, this type of model emphasizes the relationship between growth rates and R&D input endowment. In the first generation models of R&D-based growth (e.g. Romer 1990), human capital and labor that are used in R&D activities and endowments determine the economic growth rate, which are both given exogenously. Whereas, our assumption of capital R&D input, therefore, relates the amounts of capital to economic growth rates. Furthermore, because the capital stock is determined endogenously, our model contains the mechanism that the growth rate is determined by an endogenously-determined endowment of R&D input factors. Therefore, our results indicate that lower costs of intermediate goods, higher productivity of R&D, and a lower population growth rate are necessary to realize positive long-run growth.

Secondly, we derive some implications for the role of government in economic development. An optimal growth path is resolved and economic welfare is examined. Distortion of the intermediate goods sector causes the equilibrium capital stock to be overly small. The GDP growth rate of the present model is positively related with capital. Therefore, a smaller capital stock engenders a lower GDP growth rate. The capital stock is stimulated by interest rate subsidies. Consequently, the growth rate in a steady state is drawn up and optimal growth is realized. This optimal policy can enhance economic welfare, but it can not enable the economy to ride on a long-run growth path if the optimal path of an economy is a steady state with no growth.

The paper is organized as follows. The model is established and the conditions of a decentralized economy are derived in Section 2. Analysis of the steady state is presented in Section 3 and Section 4 describes the social planner problem and optimal policy. Section 5 presents the conclusion.

2 The Model

This study adopts a Romer-type (1990) production structure. Three sectors are used in the present analysis: final goods, intermediate goods, and R&D. Three factors are used: labor, capital, and knowledge. Final goods are consumed as consumption goods or are invested as physical capital. They are produced with labor (L), capital employed in the final goods sector (K_Y), and a cluster of intermediate goods.[2] In this study, capital can be used for final goods production (K_Y) and investment to create new varieties of capital, namely R&D activities (K_A). The market-clearing condition for capital imposes $K = K_Y + K_A$, where K is the total amount of capital in the economy. The labor force is supplied inelastically with growth at a constant exogenous rate n. Labor is employed only in the final goods sector.

The production function of final goods is specified as

$$Y = L^{1-\alpha} \int_0^A x(i)^\alpha di, \qquad 0 < \alpha < 1,, \tag{1}$$

where Y, L, A, and $x(i)$ respectively indicate the final goods product, labor, the number of varieties, and i's intermediate goods inputs. Intermediate goods are produced using physical capital and are used in the final goods production process. One unit of intermediate goods is assumed to be produced by η units of capital. Therefore capital devoted to final goods production K_Y is quantified as

$$K_Y \equiv \int_0^A \eta x(i) di. \tag{2}$$

An assumption of symmetric equilibrium regarding intermediate goods, that is $x = x(i)$, makes (2) into $K_Y = \eta A x$ or $x = (1/\eta)(K_Y/A)$. Substituting $x(i) = x = (1/\eta)(K_Y/A)$ into (1) allows the following derivation:

$$Y = \eta^{-\alpha} L^{1-\alpha} A^{1-\alpha} K_Y^\alpha. \tag{3}$$

Because of the assumption that final goods Y are consumed or invested and Eq. (3), the following resource constraint of final goods holds:

$$\dot{K} = \eta^{-\alpha} L^{1-\alpha} A^{1-\alpha} K_Y^\alpha - C(= Y - C),. \tag{4}$$

The per-capita resource constraint is written as

$$\dot{k} = \eta^{-\alpha} k_Y^\alpha A^{1-\alpha} - c - nk(= y - c - nk), \tag{5}$$

where the per-capita value of variable Z is written as $z(\equiv Z/L)$. Differentiating Y in Eq. (1) with respect to time yields $g_Y = (1-\alpha)n + (1-\alpha)g_A + \alpha g_K$, where g_Z is the growth rate of a variable Z. That is, $g_Z \equiv \dot{Z}/Z$. Equation (1) further implies that $g_Y = g_K = g_C$ in a steady state. These two conditions

[2] The scale of the cluster, that is, the variety of intermediate goods (A), can be regarded as technological stock in this economy.

produce the following relation in a steady state $g^* \equiv g_Y = g_C = g_K = n + g_A$. It is sometimes convenient to consider the variables of a constant in steady states. Therefore, we define the knowledge-adjusted per-capita value of a variable Z as $\tilde{z}(\equiv Z/(AL))$.

2.1 Decentralized Economy

The final goods sector is competitive, Eq. (1) yields the first order conditions (FOCs) of final goods production are given as $\frac{\partial Y}{\partial L} = w$, and $\frac{\partial Y}{\partial x(i)} = p(i)$, where w and $p(i)$ are the real wage and the price of ith sector intermediate goods, respectively.

Designs of intermediate goods are protected by patents. Therefore, intermediate goods are supplied monopolistically. In addition, a firm with a patent of ith intermediate goods production can be designated as a ith intermediate goods firm. As stated earlier, it is assumed that one unit of intermediate goods is produced using η units of capital. The profit of ith intermediate goods sector is given as $\pi(i) \equiv p(i)x(i) - r\eta x(i)$, where r is the rental price of capital and $\pi(i)$ is the profit of the ith intermediate goods firm. The intermediate goods firm maximizes this profit subject to $\frac{\partial Y}{\partial x(i)} = p(i)$. This optimization provides the following:

$$x(i) = x = \left(\frac{\alpha^2}{r\eta}\right)^{\frac{1}{1-\alpha}} L, \qquad p(i) = p = \left(\frac{\eta}{\alpha}\right) r.$$

From Eqs. (1), (2) and FOCs, the market prices are obtained as

$$w = (1-\alpha)\frac{Y}{L}, \quad r = \alpha^2 \frac{Y}{K_Y}, \quad \text{and} \quad \pi = \pi(i) = \alpha(1-\alpha)\frac{Y}{A}. \tag{6}$$

Innovation is assumed to be the discovery of a new design of intermediate goods that are added to the existing set of intermediate goods, therefore the increment of new variety is the time differentiation of knowledge $\dot{A}$. In the process of innovation, since it is assumed that the input is capital, firms undertake R&D by paying the rental cost r. R&D firms create the designs of new intermediate goods, and the patents of these designs bear the stream of monopoly profits π. The present value of this stream represents the value of R&D:

$$v \equiv \int_0^\infty \pi^M(\tau) e^{-\int_0^\tau r(s)ds} d\tau.$$

Thus, aggregate revenue and cost of R&D are respectively given as $v\dot{A}$ and rK_A.

Free entry of R&D is assumed, then if the profit of R&D is larger than the cost of R&D, then an infinite amount of capital would be allocated to R&D activities; therefore, this cannot hold in equilibrium. On the other hand, if the profit of R&D is less than the cost of R&D, then investment in R&D is unprofitable, and no resource is allocated to R&D and an equilibrium without R&D ($K_A = 0$) occurs. Thus, when the economy is in equilibrium with positive R&D acitivities, then the revenue of R&D must equate the cost of R&D. Thus, the relationships between market equilibrium and capital allocation are summarized as

$$\left.\begin{array}{l} K_A = 0 \\ K_A > 0 \\ \text{Not in equilibrium} \end{array}\right\} \Longleftrightarrow rK_A \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} v\dot{A}, \tag{7}$$

If R&D is undertaken, technological knowledge is assumed to increase according to

$$\dot{A} = \delta k_A = \delta(1-u)k. \tag{8}$$

Namely, the increment of knowledge linearly depends on the per capita capital devoted into the R&D activities. The essentially same type of R&D function is adopted in Grossman and Heipman (1991, Ch3) and Funke and Strulik (2000). Differentiating v with respect to time, we obtain the following asset equations:

$$rv = \pi + \dot{v} \quad \text{or} \quad v = \frac{\pi}{r - g_v}. \tag{9}$$

We examine the consumption decision to close the model. It is assumed that a representative household maximizes an additively separable utility function subject to a budget constraint:

$$U_t = \int_t^\infty \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma} e^{-\rho(\tau - t)} d\tau, \qquad \sigma > 0, \tag{10}$$

subject to

$$\dot{k} = rk + w + \frac{\pi A}{L} - c - nk, \tag{11}$$

where c, ρ, n, σ, k, and $\frac{\pi A}{L}$ respectively denote the per-capita consumption, discount rate, population growth rate, constant relative risk aversion (CRRA), per-capita capital holding, and per-capita dividend of the R&D

firm. An optimal condition regarding consumption is the Keynes-Ramsey rule,

$$\sigma \frac{\dot{c}}{c} = r - n - \rho, \tag{12}$$

and the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \lambda_t k_t = 0, \tag{13}$$

where $\lambda (\equiv c^{-\sigma})$ is a shadow price of per-capita capital stock k.

2.2 R&D Activity Level and Growth Rates in Steady States

Next, assume that all variables in the model grow at constant rates (possibly zero). That is, the economy is in a steady state. In such a case, the equilibrium allocation of capital is considered as follows. Substituting the arbitrage equation for R&D Eq. (9),

From $r - g_v = (\pi/v)$ (from the arbitrage equation for R&D Eq. (9)), and $v = (K_A r)/\dot{A}$ (from the definition of v in Eq. (7)), it follows that: $r - g_v = (\dot{A}\pi)/(rK_A)$. Substituting $\pi = \alpha(1-\alpha)(Y/A)$, $r = \alpha^2(Y/K_Y)$ (from Eq. (6)), and $g_A = \delta k_A$ into the above equation $r - g_v = (\dot{A}\pi)/(rK_A)$, we obtain the arbitrage equation of capital allocation in a steady state as

$$\left.\begin{array}{l} K_A^* = 0 \\ K_A^* > 0 \end{array}\right\} \Longleftrightarrow r - g_v \left\{\begin{array}{c} > \\ = \end{array}\right\} \frac{1-\alpha}{\alpha} deltak_Y. \tag{14}$$

Equation (5) implies that $g_Y = g_K = g_C$ in a steady state. Eq. (3) yields the relationship of $g_Y = (1-\alpha)n + (1-\alpha)g_A + \alpha g_K$. These two equations yield the condition of $g^* \equiv g_Y = g_K = g_C = n + g_A$. Therefore, $g_y = g_k = g_c = g_A$. Substituting $g_c = g_A$ into the Keynes-Ramsey rule of Eq. (12), we obtain those equations in a steady state as

$$\sigma g_A = r - \rho - n. \tag{15}$$

In a steady state, $g_v = r - \frac{\pi}{v}$ implies $g_v = g_\pi$. From this fact and $g_Y = g_A + n$ and $g_\pi = g_Y - g_A$ (from Eq. (6)), the following equation must hold:

$$g_v = g_\pi = g_Y - g_A = n. \tag{16}$$

Substituting Eqs. (15) and (16) into the equality in Eq. (14), we obtain a condition for R&D:

$$\left.\begin{array}{c}\text{No R\&D}\\ \text{Positive R\&D}\end{array}\right\} \Longleftrightarrow \rho + \sigma g_A \left\{\begin{array}{c} > \\ = \end{array}\right\} \frac{1-\alpha}{\alpha} k_Y. \tag{17}$$

Whether the economy conducts R&D or not depends on Eq. (17).

From (8), the growth rate of knowledge is given as

$$g_A = \delta \tilde{k}_A = \delta(\tilde{k} - \tilde{k}_Y), \quad \text{for} \quad \delta > 0. \tag{18}$$

Equation (18) derives the following from Eq. (17):

$$\rho + \sigma\delta(\tilde{k}^D - \tilde{k}_Y^D) = \delta \frac{1-\alpha}{\alpha} \tilde{k}_Y^D, \tag{19}$$

where $\tilde{k}^D$ is the amount of steady state knowledge-adjusted per-capita capital in a decentralized economy. This equation stems from the arbitrage equation for capital allocation between final goods production and R&D activities and determines the equilibrium capital allocation. Solving this equation with respect to u, we obtain the equilibrium capital allocation as

$$k_Y^D = \frac{\rho + \sigma\delta\tilde{k}^D}{\left(\sigma + \frac{1}{\alpha} - 1\right)\delta}. \tag{20}$$

The value of $\tilde{k}_Y^D$ determines the intensity of R&D activities. Therefore, it determines the growth rate of the economy in a steady state. Because $\frac{1}{\alpha} - 1 > 0$, $\tilde{k}_Y^D$ is always positive. However, $\tilde{k}_Y^*$ might be larger than $\tilde{k}$. If $0 < \tilde{k}_Y^D < \tilde{k}$, the economy's growth rate is given as $\delta(\tilde{k}^D - \tilde{k}_Y^D)) > 0$. The condition for this case is given as

$$\tilde{k}^D > \frac{\alpha\rho}{(1-\alpha)\delta} (\equiv \underline{\tilde{k}}). \tag{21}$$

If $\tilde{k}_Y^D \geq \tilde{k}^D$, equilibrium value of $tildek_Y^*$ is $\tilde{k}$ and the growth rate of the economy is 0. To sum up:

Lemma 1: *Whether a decentralized economy has a steady state of positive R&D or no R&D is dependent on the endowment of the steady-state capital stock $\tilde{k}^D$. The condition is given as*

$$\tilde{k}^D \left\{\begin{array}{c} > \\ < \end{array}\right\} \underline{\tilde{k}} \Leftrightarrow \left\{\begin{array}{l} \text{Positive R\&D Steady State} \\ \text{No R\&D Steady State} \end{array}\right. \tag{22}$$

This Lemma implies that a sufficient (effective) endowment of factor input for R&D is necessary for steady states with positive R&D activities. Therefore, the amount of capital is an important determinant of the steady state in the present paper.

Then, substituting Eq. (20) into Eq. (18), the growth rate of the economy is given as

$$g(k^D) = \frac{\left(\frac{1}{\alpha} - 1\right)\delta\tilde{k}^D - \rho}{\sigma + \frac{1}{\alpha} - 1} \left(= \frac{\delta\tilde{k}^D - \frac{\alpha}{1-\alpha}\rho}{\frac{\alpha}{1-\alpha}\sigma + 1} \right). \tag{23}$$

It is noteworthy that this growth rate closely resembles that of Eq. (13) in Romer (1990). Both equations hold that a higher R&D efficiency δ and amount of R&D input (knowledge-adjusted per-capita capital stock $\tilde{k}$ in the present study, and a higher human capital amount in Romer 1990) raises the economic growth rate. Conversely, a smaller subjective discount rate ρ and a smaller CRRA parameter σ increase the economic growth rate. We have set R&D input as capital. Therefore, we can endogenously derive the capital endowment.

Finally in this section, we derive a transversality condition in the steady state. Substituting $g_k = g_c = g^*$ (from the condition of the steady state), $g_\lambda = -\sigma g_c$ (from the definition of λ) and Eq. (23) into the transversality condition Eq. (13), we obtain the transversality condition in the steady state as

$$\rho > (1-\sigma)(1-\alpha)\delta\tilde{k}^D. \tag{24}$$

We assume that this condition is satisfied in the following part of this study.

3 Steady States and Tradition Paths

This section presents analysis of the properties of steady states. Our model includes two types of steady states. One is a steady state with steady growth; the other is one of no growth – a poverty trap. These steady states are derived in Subsection 3.1. Subsection 3.2 analyzes the stability of the economy and depicts economic paths of development.

3.1 Two types of Steady State

3.1.1 Properties of Steady Growth Equilibrium

This section presents an exploration of the steady state with R&D, called a "steady growth equilibrium" (SGE), which implies $K_A > 0$. Therefore, $g_A > 0$ in the steady state. We have two conditions of SGE: that given by the Keynes-Ramsey rule (15), and that for R&D (17). First we derive the equilibrium capital stock. Substituting K_Y^* and g^* into the steady state Keynes-Ramsey rule (15), we obtain the following equation, which provides the value of knowledge-adjusted per-capita capital in a steady state:

$$(L(\tilde{k}^D;\alpha,\delta,\rho,n,\sigma)\,\text{or}\,L(\tilde{k}^D)\equiv)n+\frac{\frac{1}{\alpha}-1}{\sigma+\frac{1}{\alpha}-1}\rho+\frac{\sigma\left(\frac{1}{\alpha}-1\right)}{\sigma+\frac{1}{\alpha}-1}\delta\tilde{k}^D$$
$$=\alpha^2\eta^{-\alpha}\left(\frac{\sigma+\frac{1}{\alpha}-1}{\frac{\rho}{\delta}+\sigma\tilde{k}^D}\right)^{1-\alpha}(\equiv R(\tilde{k}^D;\alpha,\delta,\eta,\rho,\sigma)\,\text{or}\,R(\tilde{k}^D)). \quad (25)$$

Therein, $L(\tilde{k}^D)$ is an upward sloping line and $R(\tilde{k}^D)$ is a curve that decreases from $\lim_{\tilde{k}\to-\rho/(\delta\sigma)}R(\tilde{k}^D)=\infty$ to $\lim_{\tilde{k}^D\to\infty}R(\tilde{k}^D)=0$. These two equations are portrayed graphically in Fig. 1.

The properties of a steady growth path in a decentralized economy require the following equation.

$$\text{Steady Growth Equilibrium (SGE):}\quad \tilde{k}^*=arg\{\tilde{k}^D:L(\tilde{k})=R(\tilde{k}^D)\}, \quad (26)$$

where $\tilde{k}^*$ is the knowledge-adjusted per-capita capital stock in a steady growth path. Differentiating $L(\tilde{k}^*;\alpha,\delta,\rho,n,\sigma)$ and $R^D(\tilde{k}^*;\alpha,\eta,\delta,\rho,\sigma)$ with respect to $\tilde{k},\delta,\eta,n$ and ρ, we obtain the following results; $L_{\tilde{k}}>0$, $L_\delta>0$, $L_\eta=0$, $L_\rho>0$, $L_n>0$, $R_{\tilde{k}}<0$, $R_\delta>0$, $R_\eta<0$, $R_\rho<0$, and $R_n=0$. Therefore, these derivatives yield the following:

$$\frac{d\tilde{k}^*}{d\eta}<0,\quad \frac{d\tilde{k}^*}{d\rho}<0,\quad \text{and}\quad \frac{d\tilde{k}^*}{dn}<0. \quad (27)$$

The sign of $\frac{d\tilde{k}^*}{d\delta}$ is ambiguous.

Secondly, it must satisfy the R&D condition for the SGE. From Eq. (22), the condition of positive growth is given as $\tilde{k}^*>\underline{\tilde{k}}$. Because $L(\tilde{k})$ and $R^D(\tilde{k})$ respectively indicate monotonous increasing and decreasing functions, the condition $\tilde{k}^*>\underline{\tilde{k}}$ is equivalent to

$$L(\underline{\tilde{k}})<R(\underline{\tilde{k}}). \quad (28)$$

Substituting $\underline{\tilde{k}}$ into $L(\tilde{k})$ and $R(\tilde{k})$, we obtain the condition of SGE as the following inequality:

$$\text{Parameter Condition of SGE:} \quad n + \rho < \Omega(\rho; \alpha, \eta, \delta), \tag{29}$$

where $\Omega \equiv \alpha^{1+\alpha}\eta^{-\alpha}\left(\frac{\delta(1-\alpha)}{\rho}\right)^{1-\alpha}$. Small n, small ρ, small η and large δ realize this condition.

Uniting conditions (27) and (29) strongly indicates that lower costs of intermediate goods, lower subjective discount rates and lower population growth rates are necessary to obtain a larger knowledge-adjusted per-capita capital stock.

3.1.2 Properties of a No-Growth Equilibrium

Next, we study the steady state without R&D, called "no growth equilibrium" (NGE), which implies $\tilde{k}_Y = \tilde{k}$. Therefore, $g_A = 0$ in the steady state. Eq. (5) and $g_A = 0$ imply that the growth rate in this case is given as $g_K = g_Y = g_C \equiv g^{**} = n$. Equation (6), $g^{**} = n$ and $g_A = 0$ imply that $g_V = g_\pi = g_Y - g_A = n$. Substituting $g_c = 0$ into Eq. (12) yields $r = \rho + n$. Equation (6) gives $r = \alpha^2\eta^{-\alpha}\tilde{k}^{\alpha-1}$, and Eq. (12) and $g_c = 0$ yield $r - n - \rho = 0$. From these two equations, equilibrium capital stock in this case is given as $r = \alpha^2\eta^{-\alpha}\tilde{k}^{D\alpha-1} = n + \rho$. Solving this equation, we obtain the steady state of NGE as

$$\text{No Growth Equilibrium:} \quad \tilde{k}^{**} = \left[\frac{\alpha^2\eta^{-\alpha}}{n+\rho}\right]^{\frac{1}{1-\alpha}}, \tag{30}$$

where $**$ implies the steady-state values of NGE. Equation (30) implies that larger n and ρ bear less equilibrium capital stock. Substituting equilibrium capital stock in poverty traps Eq. (30) into the necessary condition $\tilde{k}^{**} < \underline{\tilde{k}}$ (from Eq. (22)), we obtain the condition of an NGE given as

$$\text{Parameter Condition of an NGE:} \quad n + \rho > \Omega(\rho; \alpha, \eta, \delta). \tag{31}$$

Large n, large ρ, and small δ realize this case.

3.1.3 Determination of SGE or NGE

The condition of Eq. (29) and the condition of Eq. (31) are mutually exclusive. Therefore:

Lemma 2: *An economy has a unique long-run steady state of steady growth or poverty traps that are determined by the following condition:*

$$n + \rho \left\{ \begin{array}{c} > \\ < \end{array} \right\} \Omega(\rho; \alpha, \eta, \delta) \Leftrightarrow \left\{ \begin{array}{l} \text{No-Growth Equilibrium (NGE)} \\ \text{Steady-Growth Equilibrium (SGE)} \end{array} \right. \tag{32}$$

Eq. (32) shows that the parameter set $\{n, \alpha, \eta, \rho, \delta\}$ *uniquely determines either steady state. Thus, deep parameters determine the growth rate of long-run growth rate.*

These relationships are drawn in Fig. 2. As shown there, $n + \rho$ is a linear increasing function and Ω is a monotonously decreasing function and $\lim_{\rho \to 0} \Omega(\rho) = \infty$; both lines have only one solution written as $\underline{\rho}$. This $\underline{\rho}$ determines the upper bound of the subjective discount rate ρ for realization of SGE. If ρ is larger than $\underline{\rho}$, the economy is caught in an LNP. If ρ is smaller, the economy has an LGP. The increase of the population growth rate n shifts $n + \rho$ upward and Ω is invariant for the change of n. For that reason, the increase of n decreases the value of $\underline{\rho}$. From the definition of Ω, $\Omega_\eta < 0$, and $\Omega_\delta > 0$, an increase of η and a decrease of δ decrease the value of $\underline{\rho}$. Therefore, in the case where n and η increase and δ decreases, a country with a given ρ becomes hard-pressed to satisfy the conditions of LGP.

Furthermore, regarding the relationship between population growth and the per-capita GDP growth rate, because $dg(\tilde{k}^*)/d\tilde{k}^* < 0$ (from Eq. (27)) and $d\tilde{k}^*/dn < 0$ (from Eq. (23)) hold in SGE, population growth is negatively related to economic growth. Whereas, in the NGE, the per-capita GDP growth rate is 0. Therefore, our model implies either negative or no correlation between the per-capita income growth and population growth. It is well established in the literature that results of semi-endogenous growth models conflict with empirical studies. Studies, for example, by Kelley (1988), Kelley and Schmidt (1995), and Ahituv (2001), report weak negative correlation between the per-capita income growth and the population growth. Consequently, our study is more acceptable according to the empirical findings.[3]

[3]From the viewpoint of eliminating this defect of semi-endogenous growth model, Strulik (2005) developed the model with R&D, human capital accumulation and population growth, which shows an ambiguous relationship between endogenous growth and population growth.

3.2 Stability of Steady States and Transitional Paths

In a steady growth path, the decentralized economic system comprises $\dot{L} = nL$, Eqs. (4), (9), (7), (12), and (18). Defining $\tilde{k}_Y \equiv u\tilde{k}$, we re-compose the system into that constituted by $\tilde{k}$, $\tilde{c}$ and $\tilde{k}_Y$ (see Appendix A1 for detailed derivation of the following section of this paper).

Consider linearization of a system composed of Eqs. $\tilde{k}$ (Eq. (43) in A1), $\tilde{c}$ (Eq. (44) in A1), and $\tilde{k}_Y$ (Eq. (45) in A1). In the steady-growth equilibrium, the Jacobian of this linearized system J^* is given as

$$J^* = \begin{pmatrix} -(n + 2\delta\tilde{k}^* - \delta\tilde{k}^*_Y) & -1 & \alpha\eta^{-\alpha}\tilde{k}_Y^{*\alpha-1} + \delta\tilde{k}^* \\ -\delta c^* & 0 & \frac{1}{\sigma}\left(\alpha^2(\alpha-1)\eta^{-\alpha}\tilde{k}_Y^{*\alpha-2} + \sigma\delta\right)c^* \\ 0 & 0 & \frac{2-\alpha}{\alpha}\delta\tilde{k}^*_Y + n \end{pmatrix}.$$

From Appendix A1, the characteristic equation $|J^* - Iq| = 0$ has two positive and one negative roots. This result provides the following results of the transition path. An SGE is saddle-stable. It has a unique trajectory that converges to an SGE.

For the no-growth equilibrium, we can obtain the Jacobian of the no-growth equilibrium J^{**} by substituting $\tilde{k}^*_Y = \tilde{k}^* = \tilde{k}^{**}$ into J^*;

$$J^{**} = \begin{pmatrix} -(n + \delta\tilde{k}^{**}) & -1 & \alpha\eta^{-\alpha}\tilde{k}^{**\alpha-1} + \delta\tilde{k}^{**} \\ -\delta\tilde{c}^{**} & 0 & \frac{1}{\sigma}\left(\alpha^2(\alpha-1)\eta^{-\alpha}\tilde{k}^{**\alpha-2} + \sigma\delta\right)\tilde{c}^{**} \\ 0 & 0 & \frac{2-\alpha}{\alpha}\delta\tilde{k}^{**} + n \end{pmatrix}.$$

As shown in Appendix A1, the system with Jacobian J^{**} also has saddle-stable property: it has a unique trajectory that converges to a no-growth equilibrium.

These stability analyses present the following conclusion: Both steady states of SGE and NGE have unique saddle-stable paths. We respectively designate the steady state and the transition path convergences of SGE and NGE as the "long-run (positive) growth path" (LGP), and "long-run no-growth path"(LNP).

The phase diagrams of these two types of steady states are drawn together in Fig. 3. Even if two economies have similar initial endowments, they present the possibility of having opposite growth experiences if they have different steady state types given by Lemma 2. Therefore, economies are polarized into two types of steady states: an SGE with high knowledge-adjusted per-capita capital and an NGE with low knowledge-adjusted per-capita capital. In summary, we obtain the following proposition of decentralized economy:

Proposition 1 *An economy has a unique steady state and a perfect foresight saddle-stable transition path that is convergent with the steady state. The long-run growth phase, showing either steady growth or poverty traps, is determined uniquely according to technological parameters (α, η and δ) and preference and population parameters (ρ and n).*

4 Optimal Growth and Economic Policy

The previous section shows the ultimate results regarding economic growth. Following the perfect foresight path determined by given parameters, an economy is convergent to a steady state with long-run growth or no growth. However, the role of the government in economic growth and development has been emphasized in many studies. Therefore, this section examines the possibilities of the role of government through economic policies on the economic development.

4.1 Command Economy

To obtain the welfare properties of decentralized solution, consider the social planner problem of this growth model (see Appendix A2 for detailed derivation of the following part). A benevolent government is assumed to maximize the representative household's utility function Eq. (10). Therefore, a Hamiltonian of the government is written as

$$\mathcal{H} = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda(\underbrace{\eta^{-\alpha} k_Y^{\alpha} A^{1-\alpha}}_{y} - c - nk) + \mu \underbrace{\delta(k - k_A)}_{\dot{A}},$$

where λ and μ respectively represent the shadow price of per-capita capital and knowledge. Solving this social planning problem (see Appendix A2 for a detail derivation), we obtain the optimal growth rate of the economy as

$$g^*(\tilde{k}^{op}) = \frac{\left(\frac{1}{\alpha} - 1\right) \delta \tilde{k}^{op} - \rho}{\sigma + \frac{1}{\alpha} - 1}, \tag{33}$$

where $\tilde{k}^{op}$ is the knowledge-adjusted per capita capital stock in the steady state of the command economy. g^* posits that a higher subjective discount rate ρ, a higher R&D efficiency δ, a knowledge-adjusted per-capita capital stock $\tilde{k}$, and a smaller CRRA parameter σ raise the growth rate of an economy.

From Eq. (33), the condition of feasible long-run positive growth $g^*(\tilde{k}^{op}) > 0$ is given as

$$\tilde{k}^{op} > \underline{\tilde{k}}. \tag{34}$$

Uniting Eq. (21) and (34), we find that critical value of positive growth rate in a command economy is identical with that of decentralized economy.

In a command economy, the following equation provides the steady-state knowledge-adjusted per-capita capital $\tilde{k}^{*op}$ as

$$\tilde{k}^{*op} = arg\{\tilde{k}^{op} : L(\tilde{k}; \alpha, \delta, \rho, n, \sigma) = R^{op}(\tilde{k}^{op}; \alpha, \eta, \delta, \rho, \sigma)\}, \tag{35}$$

where $L(\tilde{k}^{op}; \alpha, \delta, \rho, n, \sigma)$ is identical to the LHS of Eq. (25) ($\tilde{k}^D$ is replaced by $\tilde{k}^{op}$ in this section) and $R^{op}(\tilde{k}^{op}; \alpha, \eta, \delta, \rho, \sigma)$ is $1/\alpha$ times for RHS of Eq. (25); $R^D(\tilde{k}; \alpha, \eta, \delta, \rho, \sigma)$[4]. The difference between a decentralized economy and a command economy is that R^D is α times R^{op}. As a result, the steady-state properties are mutually similar. In a steady state of an optimal economy, lower ρ, η, and n produce a higher per-capita GDP growth rate. These are common to the steady state in the decentralized economy. However, the equilibrium capital stock of the command economy is larger than that of the decentralized economy: the growth rate of this economy is higher than that of decentralized economy.

That argument implies that sufficient capital is necessary for steady growth. No R&D investment is optimal if this condition is lacking; consequently, $K_A = g_A = 0$, and the poverty-trap knowledge-adjusted per-capita capital stock in a command economy $\tilde{k}^{**op}$ are given as

$$\tilde{k}^{**op} = \left[\frac{\alpha\eta^{-\alpha}}{n+\rho}\right]^{\frac{1}{1-\alpha}}. \tag{36}$$

From $L(\tilde{k})$ and $R^{op}(\tilde{k})$ (and see Fig. 1), the condition Eq. (34) is equivalent to

$$L(\underline{\tilde{k}}) < R^{op}(\underline{\tilde{k}}). \tag{37}$$

[4] R^{op} is also drawn in Figure 1.

Substituting $\underline{\tilde{k}}$ into $L(\tilde{k})$ and $R(\tilde{k})$, we obtain the inequality,

$$n + \rho < \Omega^{op}(\rho; \alpha, \eta, \delta), \tag{38}$$

where $\Omega^{op} \equiv \alpha^{\alpha}\eta^{-\alpha}\left(\frac{\delta(1-\alpha)}{\rho}\right)^{1-\alpha}\left(= \frac{1}{\alpha}\Omega(\rho)\right)$. Small n, small ρ and large δ realize this case. Uniting Eqs. (35) and (36), we obtain the equilibrium capital stock in command economy $\tilde{k}^{op}$ as

$$\tilde{k}^{op} = \left\{ \begin{array}{ll} \tilde{k}^{*op} & \text{if} \quad n+\rho > \Omega^{op} \\ \tilde{k}^{**op} & \text{if} \quad n+\rho < \Omega^{op} \end{array} \right\}$$
$$\Rightarrow \text{Long-run optimal growth is} \left\{ \begin{array}{l} \text{Steady Growth} \\ \text{No Growth} \end{array} \right. . \tag{39}$$

4.2 Effects of Economic Policies

Here, taxes and subsidies are introduced into our model. It is proposed that a constant rate tax $\tau > 0$ (a subsidy if $\tau < 0$) is levied (provided) for interest (rental price of capital) and the profit of the intermediate sector, as

$$\begin{aligned} r^{\tau} &\equiv (1-\tau_r)r = (1-\tau_r)\alpha^2 \frac{y}{k_Y}, \\ \pi^{\tau} &\equiv (1-\tau_\pi)\pi = (1-\tau_\pi)\alpha(1-\alpha)\frac{y}{A}, \end{aligned}$$

where τ_r and τ_π respectively represent an interest tax rate and a profit-tax rate.

The existence of distortion in the intermediate goods market leads the decentralized economy to accumulate less knowledge-adjusted per-capita stock of capital than the command economy. For that reason, an economic policy to promote capital accumulation by subsidizing an interest rate always improves economic welfare.

The government is assumed to finance these subsidies using lump-sum tax revenues. The total tax revenue is expressed as T^{LS}. We presume that the government cannot borrow: the budget constraint $\tau_r K + \tau_\pi A + T^{LS} = 0$ is satisfied. Translating r and π in Eqs. (44) and (45)(in Appendix A1) to r^{τ} and π^{τ}, we determine both lines after taxation (or subsidy) as follows.

$$g_{\tilde{c}} = \frac{1}{\sigma}((1-\tau_r)\alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1} - n - \rho - \sigma\delta\tilde{k} + \sigma\delta\tilde{k}_Y), \tag{40}$$

$$g_{\tilde{k}_Y} = \frac{1}{1-\alpha}\left[\frac{\delta(1-\alpha)}{\alpha}(1-\tau_\pi)\tilde{k}_Y + n - (1-\tau_r)\alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}\right]. \tag{41}$$

For optimal growth, Eqs. (40) and (41) must correspond with Eqs. (62) and (64) (in Appendix A2), respectively.

Lemma 3: *Optimal growth rate and capital allocation are realized by the following subsidy policies:*

$$t_r^* = 1 - \frac{1}{\alpha} < 0, \quad t_\pi^* = 0. \tag{42}$$

Because $\alpha \in (0, 1)$, τ_r^* is always constant and negative, meaning that this effective policy always exists and that it increases the welfare of the economy.

4.3 "Take-off" by the Economic Policy

The previous section showed that an economic policy of interest-rate subsidies can increase the welfare of the economy. On the one hand, can an optimal subsidy policy thrust a country from a poverty trap into steady growth? The answer, at least partially, is yes.

If $\tilde{k}^{op} < \underline{\tilde{k}}$, the optimal steady state is no-growth. Therefore, the no-growth equilibrium (NGE) is an optimal path for the country. In such a case, the government must promote some parameters if they desire long-run positive growth, e.g., decreasing the population growth rate n, or increasing the R&D efficiency δ. If $\tilde{k}^{op} > \underline{\tilde{k}}$, the optimal steady state is one of positive growth. However, a decentralized economy will be caught in NGE in the long run if a country has a set of parameters that give $\tilde{k}^{**D} < \underline{\tilde{k}} (< \tilde{k}^{op})$ and the government does not execute relevant policies. In this situation, the optimal subsidy policy transforms the long-run steady state into a steady growth equilibrium (SGE). Consequently, the economy becomes able to grow permanently. A phase diagram of this case is depicted in Fig. 4. The steady state of NGE (E^{**D}) is given as the intersection of $\dot{\tilde{c}} = 0|^{NGE}$ and $\dot{\tilde{k}} = 0|^{NGE}$. If the government starts to execute the economic policy subsidizing the interest following the rule of Eq. (42), $\dot{\tilde{c}} = 0|^{NGE}$ shifts to the right side of $\underline{\tilde{k}}$ (to be $\dot{\tilde{c}} = 0|^{op}$) and $\dot{\tilde{k}} = 0|^{NGE}$ shifts lower (to be $\dot{\tilde{k}} = 0|^{op}$); the new steady state is given as E^{*op}. Corresponding to this jump of the steady state, the economy leaps to the path of the perfect foresight path converging to E^{*op}.

As discussed above, when an economy lacks the condition $\tilde{k}^{op} > \underline{\tilde{k}}$, the economy has no growth path of positive long-run growth. In such a case, the economy cannot possibly ride on a steady growth path without decreasing the welfare of the economy. Therefore some external economic aids are necessary for realizing positive long-run growth. Hence, we demonstrate effects of Official Development Assistance (ODA), which is financial or technological aid offered by developed countries to underdeveloped countries (i.e. countries in LNG) for the purpose of putting the countries on a steady growth path (i.e. countries in LPG).

Generally speaking, ODA measures such as licensing of technology and providing capital stock are respectively regarded as increments of A and K. These produce effects on the endowment of knowledge-adjusted per-capita capital $\tilde{k}$. However, the equilibrium conditions of Eqs. (30) and (31) remain wholly unaffected. Consequently, the economy will continue on a long-run no-growth path; the economy consumes capital to converge to NGE. It can be said that these effects are only to jump to a point on the long-run no-growth path. Ultimately, the steady state remains unchanged. Therefore, effective ODA to aid an economy must change a condition given Lemma 2 to change the steady state. The ODA must improve an efficiency parameter such as cost of intermediate goods production η or R&D efficiency δ. It is important for long-run growth of an independent economy that developed countries not offer stock of new technology, but instead offer the capability of creating new technology.

The above results are summarized as the following:

Proposition 2 *An interest-rate subsidy raises the welfare level through an increase in capital stock. The subsidies can increase the long-run growth rate if an optimal path of the economy is a steady growth path. The subsidy policy for interest rates enables the country to escape from poverty traps if that country has an optimal SGE path. For a country with an optimal NGE path, ODA that offers endowments such as capital stock and knowledge is ineffective for the country to ride on an optimal SGE path. If the ODA sufficiently promotes efficiency of production, for example, the cost of intermediate goods production η and R&D efficiency δ will change the long-run steady state of the economy to an SGE.*

5 Conclusion

This study developed a model with capital R&D inputs and investigated the mechanics of capital on economic growth and development. The equilibrium capital stock is positively related with the long-run growth rate. The capital stock negatively depends upon the cost of intermediate goods, a subjective discount rate, and a population growth rate. The economy has a unique steady state and a transition path converging to the steady states. For a positive steady growth, a country must have a high R&D efficiency along with a low population growth rate, low cost of intermediate goods, and a subjective discount rate.

Because the model incorporates a monopoly situation of intermediate goods production, the economy contains a distortion. For that reason, the economic policy is effective. This distortion appears in the interest rate of the economy and creates a smaller amount of long-run per-capita capital stock. Therefore, economic policies to subsidize the interest rate and increase the steady-state capital can thrust the economy onto an optimal steady growth path. When ODA is intended to aid an economy on a no-growth path to ride on a steady growyh path, aid is ineffective if it offers a stock of capital or innovative technology. Effective ODA should promote the efficiency of intermediate goods production or R&D investment: it should strengthen production and R&D activities.

Appendix

A1 Analysis of Stability

In a steady growth path, the system of the decentralized economy comprises $\dot{L} = nL$, Eqs.(4), (9), (7), (12), and (18). Defining $u\tilde{k} = \tilde{k}_Y$, we reconstruct the system into one constituted by $\tilde{k}$, $\tilde{c}$ and $\tilde{k}_Y$. Using $\tilde{k}_Y$ follows $\tilde{y} = \eta^{-\alpha}\tilde{k}_Y^{\alpha}$, and $r = \alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}$. Substituting these into Eq. (5), we obtain:

$$\dot{\tilde{k}} = \eta^{-\alpha}\tilde{k}_Y^{\alpha} - \tilde{c} - (n + \delta(\tilde{k} - \tilde{k}_Y))\tilde{k}. \tag{43}$$

Substituting $g_{\tilde{c}} + g_A = g_c$ and $r = \alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}$ into Eq. (12), we obtain the dynamics of $\tilde{c}$ as follows:

$$g_{\tilde{c}} = \frac{1}{\sigma}(\alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1} - n - \rho - \sigma\delta(\tilde{k} - \tilde{k}_Y)). \tag{44}$$

From Eqs. (7) and (18), we obtain the value of R&D as $r = \delta\frac{v}{L}$. Differentiating this equation with respect to t, we obtain $g_r = g_v - n$. Then, uniting this $g_r = g_v - n$, $g_v = r - \frac{\delta\pi}{rL}$ (from the arbitrage condition requiring Eq. (9) and free entry into R&D by Eq. (7), and by eliminating g_v from them, we obtain $g_r + n = r - \frac{\delta\pi}{rL}$. Then $r = \alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}$ yields $g_r = (\alpha - 1)g_{\tilde{k}_Y}$ and $\delta\pi/(rL) = \delta(1-\alpha)\tilde{k}_Y/\alpha$. Substitution of these equations into $g_r + n = r - \frac{\delta\pi}{rL}$ yields the dynamics of $\tilde{k}_Y$ as

$$g_{\tilde{k}_Y} = \frac{1}{1-\alpha}\left[\frac{\delta(1-\alpha)}{\alpha}\tilde{k}_Y + n - \alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}\right]. \tag{45}$$

Two types of steady states exist. One is a steady growth equilibrium (SGE) in which per-capita GDP shows positive long-run growth. The other is a no growth equilibrium (NGE) in which per-capita GDP is stationary.

If $n + \rho < \Omega$, the system has an SGE. That equilibrium is given as

$$\begin{aligned}
\tilde{k}^* &= \left(1 + \frac{1-\alpha}{\alpha}\delta\right)\frac{\tilde{k}_Y^*}{\sigma} - \frac{\rho}{\sigma\delta}, \\
\tilde{c}^* &= \frac{\delta(1-\alpha)}{\alpha^3}\tilde{k}_Y^{*2} + \left[\frac{n}{\alpha^2} - n - \frac{n+\delta}{\sigma}\left(\frac{1}{\alpha} - 1\right)\right]\tilde{k}_Y^* - \frac{n+\delta}{\sigma\delta}\rho, \\
\tilde{k}_Y^* &= \arg\left\{\tilde{k} : \frac{\delta(1-\alpha)}{\alpha}\tilde{k}_Y = \alpha^2\eta^{-\alpha}\tilde{k}_Y^{\alpha-1} - n\right\},
\end{aligned}$$

where $*$ is the index of the SGE.

The NGE of a system composed by Eqs. (43), (44) and (45) is given as

$$\tilde{k}^{**} = \left(\frac{\alpha^2\eta^{-\alpha}}{n+\rho}\right)^{\frac{1}{1-\alpha}}, \quad \tilde{c}^{**} = \eta^{-\alpha}\tilde{k}^{**\alpha} - n, \quad \text{and} \quad \tilde{k}_Y^{**} = \tilde{k}^{**},$$

where $**$ is the index of the NGE.

Consider the linearization of the system composed using Eqs. (43), (44), and (45):

$$\begin{pmatrix} \dot{\tilde{k}} \\ \dot{\tilde{c}} \\ \dot{\tilde{k}}_Y \end{pmatrix} = \underbrace{\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}}_{J^{ss}} \begin{pmatrix} \tilde{k} - \tilde{k}^{ss} \\ \tilde{c} - \tilde{c}^{ss} \\ \tilde{k}_Y - \tilde{k}_Y^{ss} \end{pmatrix}. \tag{46}$$

where $ss \in \{*, **\}$, and J^{ss} is the Jacobian of this linearized system.

The Jacobian of SGE J^* is calculated as

$$J^* \equiv \begin{pmatrix} -(n + 2\delta\tilde{k}^* - \delta\tilde{k}_Y^*) & -1 & \alpha\eta^{-\alpha}\tilde{k}_Y^{*\alpha-1} + \delta\tilde{k}^* \\ -\delta\tilde{c}^* & 0 & \frac{1}{\sigma}\left(\alpha^2(\alpha-1)\eta^{-\alpha}\tilde{k}_Y^{*\alpha-2} + \sigma\delta\right)\tilde{c}^* \\ 0 & 0 & \frac{2-\alpha}{\alpha}\delta\tilde{k}_Y^* + n \end{pmatrix}.$$

The associated characteristic equation is

$$\varphi(q) \equiv |J^* - Iq| = -q^3 + TrJ^*q^2 - BJ^*q + DetJ^* = 0,$$

where I is the identity matrix, TrJ^* is the trace of J^*, $DetJ^*$ is its determinant and

$$BJ^* \equiv \begin{vmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{vmatrix} + \begin{vmatrix} j_{22} & j_{23} \\ j_{23} & j_{33} \end{vmatrix} + \begin{vmatrix} j_{11} & j_{13} \\ j_{31} & j_{33} \end{vmatrix}.$$

The values of $Det\,J^*$, $Tr\,J^*$ and $B\,J^*$ are calculated as

$$Det\,J^* = -\delta\tilde{c}^*\left(\frac{2-\alpha}{\alpha}\delta\tilde{k}_Y^* + n\right) < 0, \tag{47}$$

$$Tr\,J^* = -2\delta\left(\tilde{k}^* - \frac{\tilde{k}_Y^*}{\alpha}\right) > 0, \tag{48}$$

$$B\,J^* = -\delta\tilde{c}^* - (n + 2\delta\tilde{k}^* - \delta\tilde{k}_Y^*)\left(\frac{2-\alpha}{\alpha}\delta\tilde{k}_Y^* + n\right) < 0. \tag{49}$$

$Tr\,J^* > 0$ is proved as follows:

Proof) Using Eq. (20) , TrJ^* is made into

$$Tr\,J^* = -2\frac{(1-\sigma)(1-\alpha)\delta\tilde{k}^* - \rho}{\alpha\sigma + 1 - \alpha}. \tag{50}$$

The assumption of satisfying the transversality condition Eq. (24) assures that the numerator of Eq. (50) is negative. Consequently, Eq. (50) is always positive. Therefore, $TrJ^* > 0$ is obtained. (Q.E.D.)

In the case of no growth equilibrium, both $\tilde{k}_Y^*$ and $\tilde{k}^*$ are made as $\tilde{k}^{**}$, and $\tilde{c}^*$ is $\tilde{c}^{**}$. Therefore, substituting $\tilde{k}_Y^* = \tilde{k}^* = \tilde{k}^{**}$ into J^*, and replacing $\tilde{c}^*$ with $\tilde{c}^{**}$, we obtain the Jacobian defined in no growth equilibrium J^{**} as

$$J^{**} \equiv \begin{pmatrix} -(n+\delta\tilde{k}^{**}) & -1 & \alpha\eta^{-\alpha}\tilde{k}^{**\alpha-1} + \delta\tilde{k}^{**} \\ -\delta\tilde{c}^{**} & 0 & \frac{1}{\sigma}\left(\alpha^2(\alpha-1)\eta^{-\alpha}\tilde{k}^{**\alpha-2} + \sigma\delta\right)\tilde{c}^{**} \\ 0 & 0 & \frac{2-\alpha}{\alpha}\delta\tilde{k}^{**} + n \end{pmatrix}.$$

It is easily verified that

$$Det\,J^{**} = -\delta\tilde{c}^*\left(\frac{2-\alpha}{\alpha}\delta\tilde{k}^{**} + n\right) < 0, \quad (51)$$

$$Tr\,J^{**} = -2\delta\left(1 - \frac{1}{\alpha}\right)\tilde{k}^{**} > 0, \quad (52)$$

$$BJ\,J^{**} = -\delta\tilde{c}^{**} - (n+\delta\tilde{k}^{**})\left(\frac{2-\alpha}{\alpha}\delta\tilde{k}^{**} + n\right) < 0. \quad (53)$$

Thus, the steady state of NGE also has a saddle stability.

Therefore, we obtain the following results for the transition path:

Proposition A1 *In both the steady state with steady growth and no growth, the system is saddle stable. It has a unique trajectory that converges to a steady growth path.*

A2 Optimal Conditions in a Command Economy

To obtain welfare properties of a decentralized solution, consider the social planner formulation of this growth model. A benevolent government is assumed to maximize the representative household's utility function Eq. (10). Therefore, a Hamiltonian of the government is written as

$$\mathcal{H} = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda(\underbrace{\eta^{-\alpha}k_Y^{\alpha}A^{1-\alpha}}_{y} - c - nk) + \mu\delta(k - k_Y),$$

where λ and μ are the respective shadow prices of per-capita capital and knowledge. Optimal conditions are obtained as

$$\lambda = c^{-\sigma} \quad (54)$$

$$\lambda\alpha\frac{y}{k_Y} = \mu\delta, \tag{55}$$

$$\rho\lambda-\dot{\lambda} = \frac{\partial\mathcal{H}}{\partial k}=-\lambda n+\mu\delta, \tag{56}$$

$$\rho\mu-\dot{\mu} = \frac{\partial\mathcal{H}}{\partial A}=\mu(1-\alpha)\frac{y}{A}, \tag{57}$$

$$\lim_{t\to\infty} e^{-\rho}\lambda_t k_t=0, \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho}\mu_t A_t=0. \tag{58}$$

Using (54), (56), and (57) derive the following equations:

$$\rho-g_\lambda = \alpha\frac{y}{k_Y}-n, \tag{59}$$

$$\rho-g_\mu = \frac{1-\alpha}{\alpha}\frac{\delta k_Y}{A}, \tag{60}$$

$$\tag{61}$$

From Eqs. (54) and (59) and using $\tilde{c}=c/A$ and $\tilde{k}=k/A$, an Euler equation is produced as

$$g_{\tilde{c}}=\frac{1}{\sigma}(\alpha\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}-n-\rho-\sigma\delta(\tilde{k}-\tilde{k}_Y)). \tag{62}$$

Because Eq. (55) yields $g_\mu+g_y=g_\lambda+g_{k_Y}$, and Eq. (59) is made into $g_\lambda=\rho-\alpha\frac{y}{k_Y}+n$, g_μ is derived as

$$g_\mu+g_y=\rho-\alpha\frac{y}{k_Y}+n+g_{k_Y}. \tag{63}$$

Substituting Eq. (63) into Eq. (60), we obtain the optimal capital allocation dynamics as: $\alpha\frac{y}{k_Y}-n-g_y+g_{k_Y}=\frac{1-\alpha}{\alpha}\delta\frac{k_Y}{A}$. Rewritten this by using the $\tilde{k}_Y$, $\tilde{y}$, $\tilde{k}$, and $g_y=(1-\alpha)g_{k_Y}+\alpha g_A$, we obtain the differential equation about $\tilde{k}_Y$ as

$$g_{\tilde{k}_Y}=\frac{1}{1-\alpha}\left[\frac{\delta(1-\alpha)}{\alpha}\tilde{k}_Y+n-\alpha\eta^{-\alpha}\tilde{k}_Y^{\alpha-1}\right]. \tag{64}$$

The system consists of these three dynamic equations: (5), (62) and (64), which imply that $g_{\tilde{k}_Y}=g_{\tilde{k}}$ and $g_A=g_k=g_c=\delta(k-k_Y)/A$ must hold in a steady state. Rewriting the system with $\tilde{k}\equiv k/A$, $\tilde{y}\equiv y/A$ and $\tilde{c}\equiv c/A$, and substituting $g_{\tilde{k}_Y}=g_{\tilde{k}}$ and $g_A=g_k=g_c=\delta(\tilde{k}-\tilde{k}_Y)$ into Eqs. (5), (62), and (64), we obtain the following equations:

$$\rho+\sigma\delta(\tilde{k}-\tilde{k}_Y) = \alpha\frac{\tilde{y}}{\tilde{k}_Y}-n, \tag{65}$$

$$\tilde{y}-\tilde{c}-(n+\delta(\tilde{k}-\tilde{k}_Y))\tilde{k} = 0 \tag{66}$$

$$\alpha\frac{\tilde{y}}{\tilde{k}_Y}+n = \frac{1-\alpha}{\alpha}\delta\tilde{k}_Y. \tag{67}$$

Eliminating $\alpha\frac{\tilde{y}}{\tilde{k}_Y} - n$ from Eqs. (65) and (67), we obtain the following equation: $\rho + \sigma\delta(\tilde{k} - \tilde{k}_Y) = \frac{1-\alpha}{\alpha}\delta\tilde{k}_Y$. Solving this equation with respect to $tildek_Y^{op}$, we obtain the optimal capital allocation on final goods production in a steady state as

$$\tilde{k}_Y^{op} = \frac{\rho + \sigma\delta\tilde{k}^{op}}{\left(\sigma + \frac{1}{\alpha} - 1\right)\delta}. \tag{68}$$

From Eqs. (18) and (68), the optimal growth rate of economy is given as

$$g^*(\tilde{k}^{op}) = \frac{\left(\frac{1}{\alpha} - 1\right)\delta\tilde{k}^{op} - \rho}{\sigma + \frac{1}{\alpha} - 1}.$$

Inserting $g^*(\tilde{k}^{op}) = g_c$ and $\tilde{k}_Y^{op}$ into Eq. (62) yields

$$n + \frac{\frac{1}{\alpha} - 1}{\sigma + \frac{1}{\alpha} - 1}\rho + \frac{\sigma\left(\frac{1}{\alpha} - 1\right)}{\sigma + \frac{1}{\alpha} - 1}\delta\tilde{k}^{op} = \alpha\eta^{-\alpha}\left(\frac{\sigma + \frac{1}{\alpha} - 1}{\frac{\rho}{\delta} + \sigma\tilde{k}^{op}}\right)^{1-\alpha}. \tag{69}$$

This equation provides the steady-state knowledge-adjusted per-capita capital $\tilde{k}^{*op}$ as

$$\tilde{k}^{*op} = arg\{\tilde{k}^{op} : L(\tilde{k}; \alpha, \delta, \rho, n, \sigma) = R^{op}(\tilde{k}^{op}; \alpha, \eta, \delta, \rho, \sigma)\}, \tag{70}$$

where $R^{op}(\tilde{k}^{op}; \alpha, \delta, \eta, \rho, \sigma)$ is the right-hand side of (69) given as

$$R^{op}(\tilde{k}^{op}; \alpha, \delta, \eta, \rho, \sigma) \equiv \alpha\eta^{-\alpha}\left(\frac{\sigma + \frac{1}{\alpha} - 1}{\frac{\rho}{\delta} + \sigma\tilde{k}^{op}}\right)^{1-\alpha} (= R^{op}(\tilde{k})).$$

and $R^{op}(\tilde{k}^{op}; \alpha, \eta, \delta, \rho, \sigma)$ is $1/\alpha$ times for RHS of Eq. (25), $R^D(\tilde{k}^{op}; \alpha, \eta, \delta, \rho, \sigma)$. $L(\tilde{k}^{op}; \alpha, \delta, \rho, n, \sigma)$ is the left-hand side of Eq. (69), which is an identical function of Eq. (25) ($\tilde{k}^D$ is replaced by $\tilde{k}^{op}$ in Eq. (69)).

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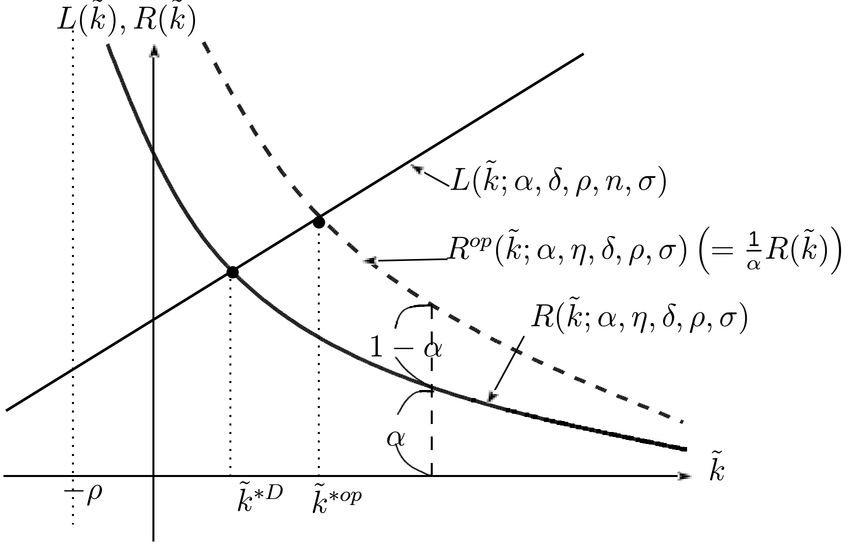

Figure 1: Equilibrium Stock of Knowledge-adjusted Per-Capita Capital in Steady Growth

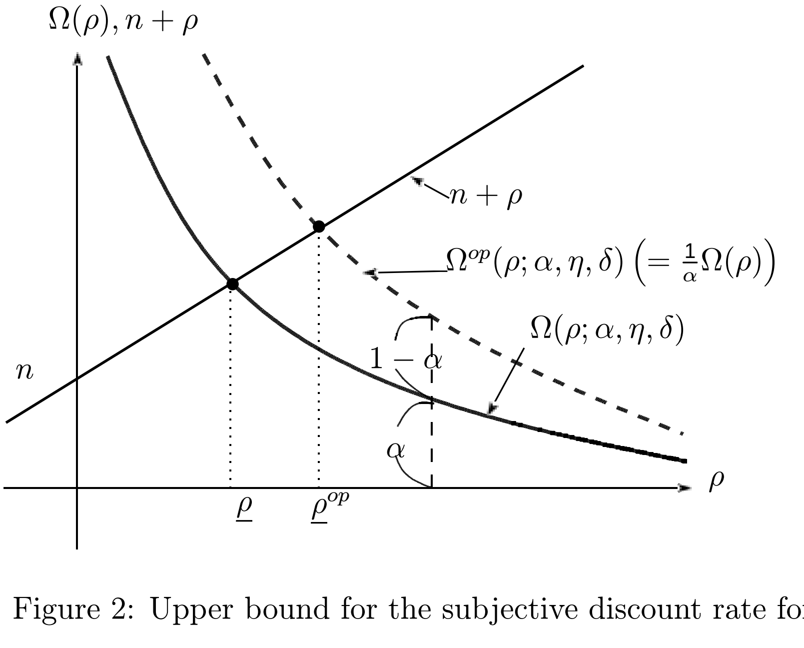

Figure 2: Upper bound for the subjective discount rate for Steady Growth

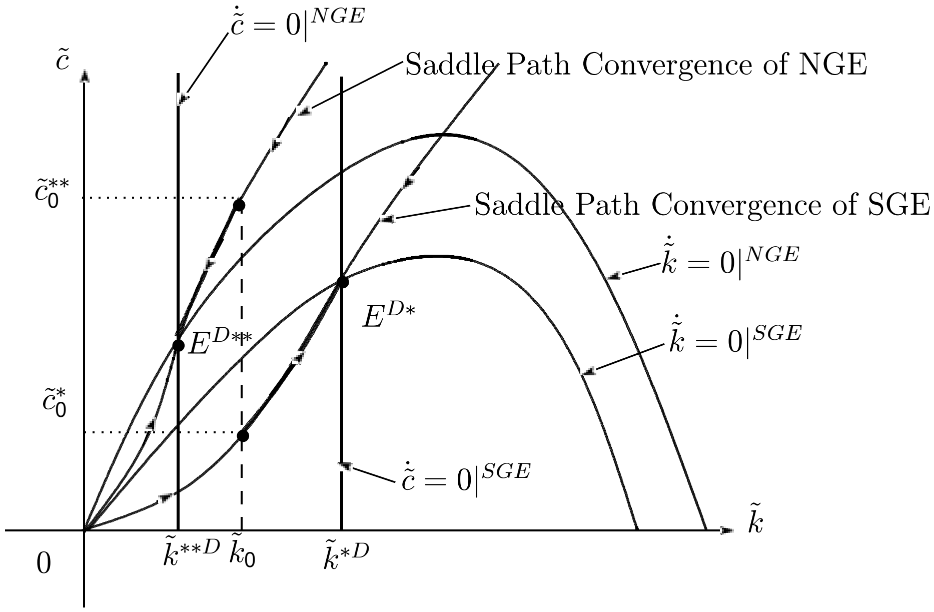

Figure 3: Polarization of Economies

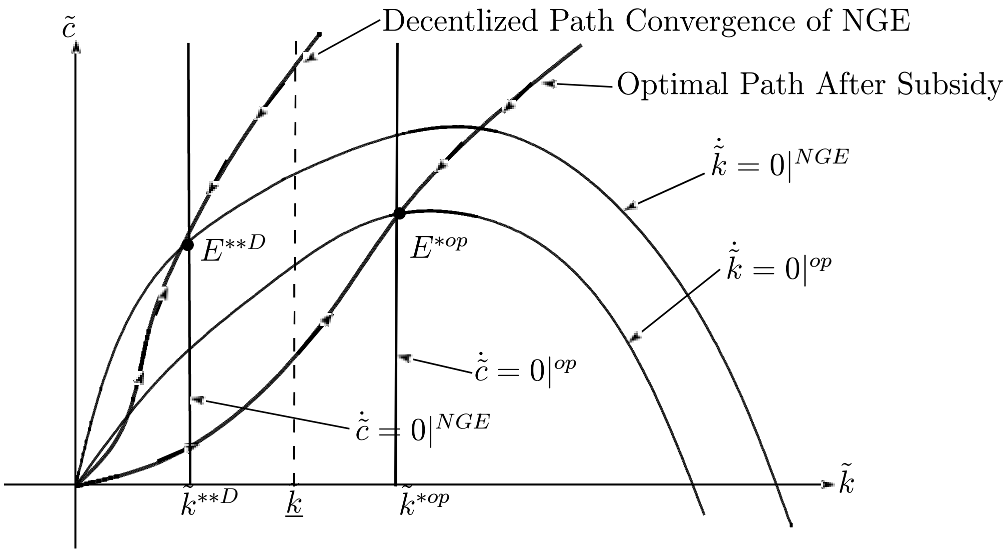

Figure 4: Take-off using an Interest-Subsidy Economic Policy