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Games in Strategic Form

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## Abstract

According to Harsanyi's representation of strategic-form games in coalitional form, the Shapley NTU values do not necessarily exist, and appear counterintuitive even in a simple example. We propose another representation, taking into account a sophisticated nature of coalition formations underlying NTU games. According to our representation, the Shapley NTU values of games in strategic form generally exist, the values satisfy a naive consistency requirement, and the number of the values is at most finite in almost every strategic-form game.

*Keywords* : Shapley NTU value, strategic-form games, consistency  
coalition structures, existence, finiteness

*JEL Classification Numbers* : C71, C72, C78

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# 1 Introduction

Cooperative game has provided many elegant theorems and significant insights. In economics, nevertheless, its predictive power is not so deserved as that of noncooperative game. It remarkably contrasts with the current prevalence and infiltration of noncooperative concepts in economists' way of thinking. One of the reasons might be that the "strategic foundations" of cooperative game have not been paid so much attention.

We will hence shed light on how to represent strategic-form games in coalitional form, studying the NTU values defined in Shapley (1969).

In the 2-person bargaining problem, we can find such a foundation in Nash (1953): prior to the actual negotiation, each player commits to a punishment to the other in case of disagreement, which is implemented in the original strategic-form game from which the bargaining problem arose.

Extending this argument to  $n$ -person games, Harsanyi(1963) proposed the following way: (1) for a list of players' actions in the original strategic-form game, let  $u(S)$  denote the sum of utilities of players in coalition  $S \subset N$ , where  $N$  is the set of all the players. (2) construct a 2-person zero-sum game  $u(S) - u(N \setminus S)$  played by coalition  $S$  and its complement  $N \setminus S$ . The strategies in the equilibrium of the zero-sum game are committed to be implemented when players in  $S$  or  $N - S$  jointly break off the negotiation.

Harsanyi's idea seems reasonable to extend the Shapley value to NTU games according to the Nash solution, since the Shapley values of players in  $S$  increases by its formula as  $u(S) - u(N \setminus S)$  increases<sup>1</sup>. However, it might not be applicable to the Shapley NTU values; they do not necessarily exist, and appear counterintuitive even in simple examples although they have a close relationship with Harsanyi values<sup>2</sup>.

We will hence propose a consistency requirement for representing strategic-form games in coalition form. The existence must be guaranteed also.

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<sup>1</sup>This is a simplification of his original NTU value defined in Harsanyi (1959). The Harsanyi value does not necessarily exists. Hart (1985) axiomatized the Harsanyi value. Owen (1972) considered another generalization of the Shapley value to NTU games.

<sup>2</sup>Since the Harsanyi value is too complicated, Shapley (1969) proposed a much simpler formulation. Aumann (1985a) axiomatized the Shapley NTU value. Imai (1983) showed a duality between the Harsanyi value and the Shapley NTU value.

It is valuable to review here the classical representations by Aumann and Peleg (1960) in the spirit of von Neumann and Morgenstern (1944).

Given a game  $G$  in strategic form, a vector  $x \in \mathbb{R}^S$  of payoffs for players in coalition  $S$  is guaranteed  $\alpha$ -effectively ( $\beta$ -effectively) in  $G$  if and only if the players in  $S$  can guarantee to themselves at least as much as  $x$ , supposing that they must announce their own joint strategy before (after) the other players outside  $S$  choose their joint strategy when players in  $S$  jointly break off the negotiation. The  $\alpha$ -effective ( $\beta$ -effective) representation of  $G$  is the NTU coalitional game  $V_c^\alpha$  ( $V_c^\beta$ ) such that

$$\begin{aligned} V_c^\alpha &= \{x \in \mathbb{R}^S \mid x \text{ is guaranteed } \alpha\text{-effectively in } G \text{ for } S\}^{\forall S \subset N} \\ V_c^\beta &= \{x \in \mathbb{R}^S \mid x \text{ is guaranteed } \beta\text{-effectively in } G \text{ for } S\}^{\forall S \subset N}. \end{aligned}$$

An important underlying assumption is: if some players in coalition  $S$  jointly broke off the negotiations in order to cooperate with members in  $S$  but without the others outside  $S$ , they should then anticipate that the others left to the negotiations would minimize their payoffs as a punishment.

They are natural extensions of von Neumann and Morgenstern's minimax criterion to NTU games, and actually useful to consider the sets of acceptable points ( $\alpha$ - and  $\beta$ -core in TU case defined in Aumann (1959)).

It might not be, however, in the best interests of the players to minimize the payoffs of the members of the deviating coalition. Rather, they would be unwilling to abandon maximizing their own payoffs. To motivate credible punishments chosen by players in each coalition, it might be worthwhile returning to such a way as Nash and Harsanyi took.

We propose a new way of representing strategic-form games in coalitional form, taking into account a sophisticated nature of events: some players can form coalition  $S$  to cooperate with the members, whereas the other players can react by forming "some" coalitions. This point is the same as in Funaki and Yamato (1999). The major difference lies in the following: the players in  $S$  then might not merge into a single one but gather as smaller subcoalitions in coalition  $S$ , as shown later with an example.

We show that our representation is well-defined, the Shapley NTU values of games in strategic form generally exist and satisfy a naive consistency, and that the number is finite in almost every strategic-form game.

The following part of this paper is organized as below. Section 2 describes the definitions of NTU games, the Shapley NTU value and strategic-form games. Section 3 shows an example where Harsanyi's representation results in a counterintuitive outcome. A consistency notion is also proposed here. Section 4 defines our representation of strategic-form games, showing that it is well-defined. Section 5 shows the existence of the Shapley NTU values of strategic-form games in our representation, referring to an algorithmic consideration. The number of the values is also studied. Section 7 closes this paper with discussions on the related literature. Funaki and Yamato (1999) is reviewed there.

## 2 Preliminaries

### The Shapley NTU Values

Let  $N = \{1, \dots, n\}$ , where  $2 \leq n < \infty$ , be the set of players. Denote by  $V$  a correspondence that assigns to each subset  $S \subset N$  a nonempty closed convex subset bounded above in  $\mathbb{R}^{|S|}$ . A game  $(N, V)$  without sidepayments is defined as a pair of  $N$  and  $V$ , also called an NTU game because of the non-transferability of utilities among players, and sometimes to simply  $V$ .

Let  $(x_i)_{i \in S} \in \mathbb{R}^{|S|}$  be a vector of payoffs for players in a subset  $S \subset N$  and  $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}_{++}^n$  be a vector of weights on players' payoffs. A  $\lambda$ -transfer game  $v_\lambda$  (with sidepayments) of an NTU game  $V$  is defined by

$$v_\lambda(S) = \max_{(x_i)_{i \in S} \in V(S)} \sum_{i \in S} \lambda_i x_i \quad \text{for each coalition } S \subset N$$

and  $v_\lambda(\emptyset) = 0$ , meaning that  $\{(x_i)_{i \in S} \in \mathbb{R}^{|S|} : \sum_{i \in S} \lambda_i x_i \leq v_\lambda(S)\}$  is the set of attainable vectors of weighted payoffs for players in  $S$ .

The solutions  $(x_i^*)_{i \in S}$  of the above maximization problem are vectors of Pareto optimal payoffs in  $V(S)$ . Varying the values of  $\lambda$ , we can derive the set of all the Pareto optimal vectors of payoffs in  $V(S)$  except extreme points that corresponds to  $\lambda_i = 0$  for some  $i \in N$ , if  $V(S)$  is convex.

Let  $\mathcal{R} : i_1, i_2, \dots, i_n$  be an ordering of players in  $N$ .  $j\mathcal{R}i$  means that  $j$  precedes  $i$  in the ordering  $\mathcal{R}$ . Denote by  $\mathcal{P}_i^{\mathcal{R}} = \{j \in N | j\mathcal{R}i\}$  the set of players who precede player  $i \in N$  in  $\mathcal{R}$ .

**Definition 1** The **Shapley value**  $\varphi(v_\lambda) = (\varphi_i(v_\lambda))_{i \in N}$  of a  $\lambda$ -transfer game  $v_\lambda$  is defined as

$$\varphi_i(v_\lambda) = \frac{1}{n!} \sum_{\mathcal{R}} v_\lambda(\mathcal{P}_i^{\mathcal{R}} \cup \{i\}) - v_\lambda(\mathcal{P}_i^{\mathcal{R}}).$$

The Shapley value  $\varphi(v_\lambda)$  is not necessarily feasible in the original NTU game  $V$ . It can be that  $\varphi(v_\lambda) \notin V(N)$ , if some vectors  $(x_i)_{i \in N}$  of payoffs on the weighted transfer hyperplane  $\{(x_i)_{i \in N} \in \mathbb{R}^n \mid \sum_{i \in N} \lambda_i x_i = v_\lambda(N)\}$  are not in  $V(N)$ . We write  $\lambda * y = (\lambda_i y_i)_{i \in N} \in \mathbb{R}^n$ , where  $y = (y_i)_{i \in N} \in \mathbb{R}^n$ .

**Definition 2** The **Shapley NTU value of a game**  $V$  are the set of vectors  $y = (y_i)_{i \in N} \in V(N)$  such that  $\lambda * y = \varphi(v_\lambda)$  for some  $\lambda \in \mathbb{R}_{++}^n$ .

In general, each NTU game has multiple Shapley NTU values. The existence is guaranteed.

## Strategic-Form Games

A *game in strategic form* is denoted as  $G = \{N, (C_i)_{i \in N}, (u_i)_{i \in N}\}$ , where  $N$  is the set of players,  $C_i$  is the set of pure strategies of player  $i$  and  $u_i : \times_{i \in N} C_i \rightarrow \mathbb{R}$  is the payoff function for player  $i$ . We hereafter restrict our attention to finite games, i.e., assume that  $|N| = n < \infty$ ,  $|C_i| < \infty \forall i \in N$ , and  $u_i(c) < \infty \forall i \in N, \forall c \in \times_{i \in N} C_i$ .

Each nonempty subset  $S \subset N$  is called a *coalition* of players. Given any game  $G$  in strategic form, a *correlated strategy* of a coalition  $S$  is any probability distribution  $\sigma_S$  over the set  $C_S = \times_{i \in S} C_i$  of possible combinations of pure strategies that players in  $S$  can choose in  $G$ . The probability of designating any  $c_S = (c_i)_{i \in S}$  is denoted by  $\sigma_S(c_S)$ . Denote by  $\Delta(C_S)$  the set of all the possible correlated strategies of  $S$ .

Denote by  $c_{-S} \in C_{-S}$  a list of pure strategies of players in coalitions other than  $S$ . We write  $\sigma_{-S} \in \Delta(C_{-S})$  for a list of correlated strategies of coalitions other than  $S$ , where  $\Delta(C_{-S})$  is the set of all the possible  $\sigma_{-S}$ .

Let  $u_i(\sigma) = u_i(\sigma_S, \sigma_{-S})$  denote the expected payoff for player  $i \in S$  when the list of correlated strategies  $\sigma = (\sigma_S, \sigma_{-S})$  is played, i.e.,

$$u_i(\sigma_S, \sigma_{-S}) = \sum_{c_S \in C_S} \sum_{c_{-S} \in C_{-S}} \sigma_S(c_S) \sigma_{-S}(c_{-S}) u_i(c_S, c_{-S}).$$

### 3 Harsanyi's Representation

Given a game  $G = \{N, (C_i)_{i \in N}, (u_i)_{i \in N}\}$  and a  $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}_{++}^n$ , a 2-person zero-sum game played by a coalition  $S$  and its complement  $N \setminus S$  is defined by the payoff function  $H_\lambda(S) : \Delta C_S \times \Delta C_{N \setminus S} \rightarrow \mathbb{R}$  such that

$$H_\lambda(S)(\sigma_S, \sigma_{N \setminus S}) = \sum_{i \in S} \lambda_i u_i(\sigma_S, \sigma_{N \setminus S}) - \sum_{j \in N \setminus S} \lambda_j u_j(\sigma_S, \sigma_{N \setminus S}),$$

which is the expected payoff paid to  $S$  by  $N \setminus S$  when a list  $(\sigma_S, \sigma_{N \setminus S})$  of correlated strategies is played. Denote by  $\text{con}A$  the convex hull of a set  $A$ .

**Definition 3** Given a game  $G = \{N, (C_i)_{i \in N}, (u_i)_{i \in N}\}$  in strategic form, the worth  $v_\lambda^H(S)$  of each coalition  $S \subset N$  is determined by equilibrium  $(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S})$  in the zero-sum game  $H_\lambda(S)$ , i.e.,

$$v_\lambda^H(S) = \sum_{i \in S} \lambda_i u_i(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S}),$$

$$\begin{aligned} \text{where } \bar{\sigma}_S &\in \arg \max_{\sigma_S \in \Delta(C_S)} H_\lambda(S)(\sigma_S, \bar{\sigma}_{N \setminus S}) \\ \bar{\sigma}_{N \setminus S} &\in \arg \min_{\sigma_{N \setminus S} \in \Delta(C_{N \setminus S})} H_\lambda(S)(\bar{\sigma}_S, \sigma_{N \setminus S}). \end{aligned}$$

Let  $V^H(N) = \text{con}\{(x_i)_{i \in N} \in \mathbb{R}^n \mid x_i \leq u_i(c), \text{ where } c \in \times_{i \in N} C_i\}$ .

**Definition 4** The **Shapley NTU values of a strategic-form game  $G$  due to Harsanyi** are the set of vectors  $\phi^H(G, \lambda) \in V^H(N)$  such that  $\lambda * \phi^H(G, \lambda) = \varphi(v_\lambda^H)$  for some  $\lambda \in \mathbb{R}_{++}^n$ .

Consider a game  $G^1$  in strategic form.  $N = \{1, 2, 3\}$ ,  $C_1 = \{t, m, b\}$ ,  $C_2 = \{L, R\}$  and  $C_3 = \{q\}$  and the payoffs are depicted as follows.

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{ccc} L & & R \end{array} \\ \begin{array}{c} t \\ m \\ b \end{array} & \left( \begin{array}{ccc|ccc} -1 & , & -1 & , & 1 & & 0 & , & -1 & , & 0 \\ -1 & , & -1 & , & -1 & & 0 & , & -1 & , & 0 \\ 0 & , & 0 & , & 0 & & 1 & , & -1 & , & 0 \end{array} \right) \end{array} \\ q \end{array}$$

Let  $\lambda = (1, 1, \mu)$  with  $\mu \in (0, 1)$ . Below is the payoff matrix of the zero-sum game played by coalitions  $\{1\}$  and  $\{2, 3\}$  generated from  $G^1$  with  $\lambda$ . The equilibrium is  $(m, Lq)$ .

$$\begin{array}{c}
Lq \qquad Rq \\
t \left( \begin{array}{cc} -\mu & 1 \\ \mu & 1 \\ 0 & 0 \end{array} \right) \\
m \\
b
\end{array}$$

Similarly computing equilibria in the other zero-sum games, we have

$$\begin{aligned}
v_\lambda^H(\{1\}) &= -1 & v_\lambda^H(\{2, 3\}) &= -1 - \mu \\
v_\lambda^H(\{2\}) &= -1 & v_\lambda^H(\{1, 3\}) &= -1 + \mu \\
v_\lambda^H(\{3\}) &= 0 & v_\lambda^H(\{1, 2\}) &= 0 & v_\lambda^H(N) &= 0,
\end{aligned}$$

and so  $\varphi_1(v_\lambda^H) = -\varphi_2(v_\lambda^H) = \mu/2$  and  $\varphi_3(v_\lambda^H) = 0$ . Since  $\phi^H(G^1, \lambda) \in V^H(N)$ , the Shapley NTU value of this game  $G$  with  $\lambda$  due to Harsanyi is

$$\phi_1^H(G^1, \lambda) = \mu/2, \quad \phi_2^H(G^1, \lambda) = -\mu/2, \quad \phi_3^H(G^1, \lambda) = 0,$$

Does player 2 really accept  $\phi_2^H(G^1, \lambda) = -\mu/2$ ? Player 3 just attends the game  $G^1$  with only one strategy, and actually receives nothing. Player 1 and 2 might then jointly propose to such a “dummy” player that they assure her of payoff  $\phi_3^H(G^1, \lambda) = 0$  but that she wait for receiving it until the conflict between them resolves with an agreement in negotiations.

If  $\phi_3^H(G^1, \lambda) > 0$ , player 1 and 2 would argue how much each one should pay to player 3, while if  $\phi_3^H(G^1, \lambda) < 0$ , player 3 might not accept such a proposal. If player 3 had more strategies, player 1 and 2 might not make such a proposal, since each one would influence player 3 to choose her strategies benevolent also to him with some sidepayments.

In this game  $G^1$ , however, it is plausible to construct such a “reduced” game  $G'$  by replacing all the payoffs of player 3 with  $\phi_3^H(G^1, \lambda) = 0$ . If  $G'$  was played, we would have  $\phi_i^H(G', \lambda) = 0 \forall i \in N$ , which suggests that player 2 reject  $\phi_2^H(G^1, \lambda) = -\mu/2$  in a real negotiation procedure.

**Definition 5** Let  $\phi$  be an NTU value of games in strategic form, and let  $G$  be a game in strategic form. A player  $i \in N$  is called a *dummy player* in  $G$  and  $\phi$ , if  $i$  has only one strategy in  $G$  and  $\phi_i(G) = 0$ . A *reduced game*  $G'$  is constructed by replacing all the payoffs for every dummy player  $i$  in  $G$  with  $\phi_i(G) = 0$ . We say that the NTU value  $\phi$  satisfies the **naive consistency** if  $\phi(G) = \phi(G')$  whenever  $G$  has a dominant strategy equilibrium.

As shown above,  $\phi^H(G^1, \lambda)$  does not satisfy the naive consistency. We should propose another way of representing any strategic-form game  $G$  in coalitional form so that the solution in our question satisfies such a weak consistency at least.

The existence must be guaranteed also, but the answer is unfortunately negative. Consider a game  $G^2$ , which is an example where the Shapley NTU value of a strategic-form game due to Harsanyi does not exist.  $N = \{1, 2\}$ ,  $C_1 = \{t, b\}$ ,  $C_2 = \{L, R\}$ , and the payoffs are depicted as follows.

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} t \\ b \end{array} & \left( \begin{array}{cc|cc} 1, & 0 & 0, & 0 \\ 0, & -1 & -1, & 0 \end{array} \right) \end{array}$$

$V^H(N) = \text{con}\{(x_i)_{i \in N} \in \mathbb{R}^2 | (x_i)_{i \in N} \leq (1, 0)\}$ . For any  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_{++}^2$ , we have  $v_\lambda^H(\{i\}) = 0$  for both  $i \in N$ , and  $v_\lambda^H(N) = \lambda_1$ . Then,  $\varphi_1(v_\lambda^H) = \varphi_2(v_\lambda^H) = \lambda_1/2$ , and so  $\phi_1^H(G^2, \lambda) = 1/2$  and  $\phi_2^H(G^2, \lambda) = \lambda_1/2\lambda_2$ . Hence,  $\phi^H(G^2, \lambda) \notin V^H(N)$ .

## 4 A New Representation

Before proceeding to the formal definition, we describe a sophisticated nature of events in our representation with the following simple TU game.

### An Example

There are 7 firms in the market, each producing  $q_i$  units of an identical commodity with the same unit cost  $c$ . The market price  $p$  is determined by  $p = a - \sum q_i$ , where  $a (> c)$  is a positive constant. Suppose that 3 coalitions form, each consisting of 4 firms, 2 firms, and 1 firm.

Firms in each coalition can decide whether to merge into a single entity or to gather as some subcoalitions in the coalition. In the market, firms in the same subcoalition cooperate to maximize the total sum of individual profits in the subcoalition, while each subcoalition competes *à la* Cournot not only with the “opponent” subcoalitions outside  $S$  but also with the other “colleague” subcoalitions in  $S$ .

The joint profit of each coalition is naturally defined as the total sum of profits of its subcoalitions. If firms merged into a single entity maximized the joint profit in every coalition, the equilibrium joint profit of each coalition would be  $((a - c)/4)^2$ . Given that firms in the other coalitions remained as a single entity, it would be best for firms in  $S^*$  to have 3 subcoalitions, since it could receive the larger equilibrium joint profits of  $3((a - c)/6)^2$ .

Then, the coalition of 2 firms would receive  $((a - c)/6)^2$ . But if it had 2 subcoalitions, it would receive a larger joint profit of  $2((a - c)/7)^2$ . Anticipating such a worst counter coalition formation by the rival firms, it is best for firms in coalition  $S^*$  to have 4 subcoalitions and receive the equilibrium joint profit of  $4((a - c)/8)^2$ , which is “guaranteed” to coalition  $S^*$ .

The key point here is that players do not necessarily cooperate even in the same coalition.

### The definition

Let  $P = [S^1, \dots, S^M]$  be a partition of the set  $N$  of players. Each nonempty subset  $S^m \subseteq N$ ,  $m = 1, \dots, M$ , is called a *coalition* of players. There can be a subpartition  $\tilde{S}^m = \{S^{1(m)}, \dots, S^{l(m)}, \dots, S^{L(m)}\}$  in each coalition  $S^m$ . Each nonempty subset  $S^{l(m)} \subseteq S^m$  is called a *subcoalition*. Denote by  $\tilde{P} = [\tilde{S}^1, \dots, \tilde{S}^M]$  a *coalition structure*<sup>3</sup>.

Each subcoalition  $S^{l(m)}$  in coalition  $\tilde{S}^m$  competes not only with “opponent” subcoalitions outside  $\tilde{S}^m$  but also with “colleague” subcoalitions in  $\tilde{S}^m$ , while players in the same subcoalition  $S^{l(m)}$  cooperatively maximize the vector  $(u_i(\sigma))_{i \in S^{l(m)}}$  of their expected payoffs, where  $\sigma = (\sigma_{S^{l(m)}}, \sigma_{-S^{l(m)}})$ .

We say that a list  $\bar{\sigma}$  of correlated strategies of subcoalitions satisfies the **best response property** under a given coalition structure  $\tilde{P}$  if, for every subcoalition  $S^{l(m)}$  in every coalition  $\tilde{S}^m \in \tilde{P}$ , there is no  $\sigma_{S^{l(m)}}$  such that  $u_i(\sigma_{S^{l(m)}}, \bar{\sigma}_{-S^{l(m)}}) > u_i(\bar{\sigma})$  for all  $i \in S^{l(m)}$ . Denote by  $\beta(\tilde{P})$  the set of such lists of correlated strategies.

Denote by  $\pi(S^m)$  the set of all the subcoalitions in  $S^m$  and by  $\tilde{\pi}(N \setminus S^m)$  the set of all the coalition structures over  $N \setminus S^m$ . We can write as  $\tilde{P} = [\pi, \tilde{\pi}]$ . Let  $x^{S^m} = (x_i)_{i \in S^m}$  be a vector of payoffs for players in coalition  $S^m$ .

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<sup>3</sup>A partition  $P$  is called a coalition structure in the standard sense.

**Definition 6** We say that  $x^{S^m}$  is  $\alpha$ -guaranteed to  $S^m$  in a strategic-form game  $G$  if and only if, for every coalition structure  $\tilde{\pi} \in \tilde{\pi}(N \setminus S^m)$  over  $N \setminus S^m$ , there exist some subcoalition  $\pi \in \pi(S^m)$  in  $S^m$  and the lists of correlated strategies  $\bar{\sigma} \in \beta(\tilde{P})$  corresponding to  $\tilde{P} = [\pi, \tilde{\pi}]$  such that

$$\min_{\bar{\sigma} \in \beta(\tilde{P})} u_i(\bar{\sigma}) \geq x_i \quad \forall i \in S^{l(m)}, \quad \forall S^{l(m)} \in \tilde{S}^m$$

An NTU game  $V^\alpha$  representing a game  $G$  in strategic form is defined as

$$V^\alpha(S^m) = \{x^{S^m} \in \mathbb{R}^{|S^m|} \mid x^{S^m} \text{ is } \alpha\text{-guaranteed to } S^m\} \quad \forall S^m \subseteq N.$$

That is, coalition  $S^m$  must announce  $\pi$  before  $\tilde{\pi}$  is chosen. In another NTU game  $V^\beta$  defined in a similar way, coalition  $S^m$  can announce  $\pi$  after  $\tilde{\pi}$  is chosen.

**Definition 7** We say that  $x^{S^m}$  is  $\beta$ -guaranteed to  $S^m$  in a strategic-form game  $G$  if and only if, for each coalition structure  $\tilde{\pi} \in \tilde{\pi}(N \setminus S^m)$  over  $N \setminus S^m$ , there exist some subcoalition  $\pi \in \pi(S^m)$  and the lists of correlated strategies  $\bar{\sigma} \in \beta(\tilde{P})$  corresponding to  $\tilde{P} = [\pi, \tilde{\pi}]$  such that

$$\min_{\bar{\sigma} \in \beta(\tilde{P})} u_i(\bar{\sigma}) \geq x_i \quad \forall i \in S^{l(m)}, \quad \forall S^{l(m)} \in \tilde{S}^m$$

An NTU game  $V^\beta$  representing a game  $G$  in strategic form is defined as

$$V^\beta(S^m) = \{x^{S^m} \in \mathbb{R}^{|S^m|} \mid x^{S^m} \text{ is } \beta\text{-guaranteed to } S^m\} \quad \forall S^m \subseteq N.$$

$\Delta(C_{S^{l(m)}})$  is the set of all the correlated strategies of a subcoalition  $S^{l(m)}$ . Denote by  $\mathcal{B}_{S^{l(m)}}(\sigma_{-S^{l(m)}})$  the set of best responses of  $S^{l(m)}$  to correlated strategies  $\sigma_{-S^{l(m)}}$  of subcoalitions other than  $S^{l(m)}$ .

**Proposition**  $\beta(\tilde{P}) \neq \emptyset$  for any coalition structure  $\tilde{P}$ . NTU games  $V^\alpha$  and  $V^\beta$  representing a game  $G$  in strategic form are then well-defined.

*Proof:* (1) Given a coalition structure  $\tilde{P}$ ,  $\Delta(C_{S^{l(m)}})$  is a  $|C_{S^{l(m)}}| - 1$  dimensional simplex in  $\mathbb{R}^{|C_{S^{l(m)}}|}$  for each subcoalition  $S^{l(m)}$ , which is compact and convex. Hence,  $\times_{S^{l(m)}} \Delta(C_{S^{l(m)}})$  is a compact and convex subset in  $\mathbb{R}^{k(\tilde{P})}$  where  $k(\tilde{P}) := \sum_{S^{l(m)}} |C_{S^{l(m)}}|$ .

(2) The vector  $(u_i(\sigma_{S^{l(m)}}, \sigma_{-S^{l(m)}}))_{i \in S^{l(m)}}$  of expected payoffs for players in  $S^{l(m)}$  is a continuous linear function in  $\sigma_{S^{l(m)}}$ . Since  $\Delta(C_{S^{l(m)}})$  is a compact convex subset in  $\mathbb{R}^{|C_{S^{l(m)}}|}$ ,  $\mathcal{B}_{S^{l(m)}}(\sigma_{-S^{l(m)}})$  is a nonempty and convex subset in  $\Delta(C_{S^{l(m)}})$ . For any  $\sigma = (\sigma_{S^{l(m)}}, \sigma_{-S^{l(m)}}) \in \times_{S^{l(m)}} \Delta(C_{S^{l(m)}})$ ,  $\mathcal{B}(\sigma) := \times_{S^{l(m)}} \mathcal{B}_{S^{l(m)}}(\sigma_{-S^{l(m)}})$  is hence non-empty and convex subset of  $\times_{S^{l(m)}} \Delta(C_{S^{l(m)}})$ . By the Berge maximum theorem,  $\mathcal{B}(\sigma)$  satisfies the upper hemi-continuity, since  $u_{S^{l(m)}}(\sigma)$  is a continuous function for each  $S^{l(m)}$ .

(3) By (1), (2) and the Kakutani fixed point theorem, there exists at least one  $\bar{\sigma} \in \mathcal{B}(\bar{\sigma})$ . The same argument applies to any coalition structures. Hence,  $\beta(\tilde{P}) \neq \emptyset$  for any  $\tilde{P}$ . Since subcoalitions are finitely many for any  $\tilde{P}$ ,  $V^\beta(S^m)$  are completely characterized, and it is straightforward that  $V^\alpha(S^m)$  and  $V^\beta(S^m)$  are closed, convex and bounded above for any  $S^m \subseteq N$ . *Q.E.D.*

## 5 The NTU Values of Games in Strategic Form

### An Algorithm

Hereafter, we refer only to  $V^\alpha$  since the same argument applies to  $V^\beta$ . To characterize  $V^\alpha$ , we first need to compute  $\bar{\sigma} \in \beta(\tilde{P})$ . Below is the algorithm that we use, which is often used in the multiobjective optimization<sup>4</sup>.

Given a  $\lambda$  and a  $\sigma$ , denote by  $u_{S^{l(m)}}^\lambda(\sigma) = \sum_{i \in S^{l(m)}} \lambda_i u_i(\sigma)$  the weighted sum of payoffs for players in  $S^{l(m)}$ . We say that a list  $\bar{\sigma}$  of correlated strategies of subcoalitions satisfies the  **$\sigma$ -best response property** under a given coalition structure  $\tilde{P}$  if, for every subcoalition  $S^{l(m)}$  in every coalition  $\tilde{S}^m \in \tilde{P}$ , there is no  $\sigma_{S^{l(m)}}$  such that  $u_{S^{l(m)}}^\lambda(\sigma_{S^{l(m)}}, \bar{\sigma}_{-S^{l(m)}}) > u_{S^{l(m)}}^\lambda(\bar{\sigma})$ . Denote by  $\beta_\lambda(\tilde{P})$  the set of such lists of correlated strategies.

$\beta_\lambda(\tilde{P}) \subseteq \beta(\tilde{P})$  for any  $\tilde{P}$ , by a well-known result in the multiobjective optimization. Since  $\mathcal{B}_{S^{l(m)}}(\sigma_{-S^{l(m)}})$  is convex as shown in (2) in the proof of Proposition, there is no “gap problem” in computing  $\sigma_{S^{l(m)}} \in \mathcal{B}(\sigma_{-S^{l(m)}})$ , and so by varying the values of  $\lambda$  we can compute all the vectors in  $\mathcal{B}_{S^{l(m)}}(\sigma_{-S^{l(m)}})$  except extreme points corresponding to  $\lambda_i = 0$ . Thus, for any  $\lambda \in \mathbb{R}_{++}^n$ , we can derive all the vectors  $\bar{\sigma} \in \beta(\tilde{P})$ .

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<sup>4</sup>See Tamura (1982) or other textbooks. Andersen and Lind (1999) showed the precise algorithm to compute the Shapley NTU value of NTU games for 2-person case.

Since  $V^\alpha(S^m)$  is closed convex and bounded above for any  $S^m$ , we can apply the same way to compute the vectors  $(x_i)_{i \in N}$  of payoffs on the weighted transfer hyperplane  $\{(x_i)_{i \in N} \in \mathbb{R}^n \mid \sum_{i \in N} \lambda_i x_i = v_\lambda^\alpha(N)\}$  with a  $\lambda$  given.

**Definition 8** The  $\alpha$ -guaranteed worth  $v_\lambda^\alpha(S^m)$  of  $S^m \subseteq N$  in a game  $G$  in strategic form with a vector  $\lambda \in \mathbb{R}_{++}^n$  of weights is defined by

$$v_\lambda^\alpha(S^m) = \max_{\pi \in \pi(S^m)} \min_{\tilde{\pi} \in \tilde{\pi}(N \setminus S^m)} F_{S^m}(\pi, \tilde{\pi}),$$

where  $F_{S^m}(\pi, \tilde{\pi}) = \min_{\bar{\sigma} \in \beta_\lambda(\tilde{P})} \sum_{S^l(m) \in \pi} u_{S^l(m)}^\lambda(\bar{\sigma})$  with  $\tilde{P} = [\pi, \tilde{\pi}]$  given. Assume that  $v_\lambda^\alpha(\emptyset) = 0$ .

Notice that  $V^\alpha(N) = \text{con}\{(x_i)_{i \in N} \in \mathbb{R}^n \mid x_i \leq u_i(c), \text{ where } c \in \times_{i \in N} C_i\}$ .

**Definition 9** The **Shapley NTU values of a game  $G$  in strategic form** are the set of vectors  $\phi^\alpha(G, \lambda) \in V^\alpha(N)$  such that  $\lambda * \phi^\alpha(G, \lambda) = \varphi(v_\lambda^\alpha)$  for some  $\lambda \in \mathbb{R}_{++}^n$ .

### The Existence

In the 2-person game  $G^2$  in section 3, there are two equilibria  $(t, L)$  and  $(t, R)$ .  $V^\alpha(N) = \text{con}\{(x_i)_{i \in N} \in \mathbb{R}^2 \mid (x_i)_{i \in N} \leq (1, 0)\}$ . In the latter case,  $\phi^H(G^2, \lambda) = (1/2, \lambda_1/2\lambda_2) \notin V^\alpha(N)$  for any  $\lambda \in \mathbb{R}_{++}^2$ , as previously shown in section 3. In the former case, we have  $\phi^H(G^2, \lambda) = (1, 0) \in V^\alpha(N)$ . Hence, there exists the Shapley NTU value of  $G^2$ .

Let  $\delta$  be any real number in the open interval  $(0, 1/n)$ . Define  $\Lambda := \{(\lambda_i)_{i \in N} \in \mathbb{R}_{++}^n \mid \sum_{i \in N} \lambda_i = 1, \lambda_i \geq \delta \forall i \in N\}$ .

**Theorem 1** *There exists at least one Shapley NTU value  $\phi^\alpha(G, \lambda)$  of a game  $G$  in strategic form.*

*Proof:* It suffices to show the existence of the NTU value for  $\lambda \in \Lambda$ .

(1) We first show that  $\varphi(v_\lambda^\alpha)$  is continuous in  $\lambda$ . Given a coalition structure  $\tilde{P} = [\tilde{\pi}, \pi]$ , let  $\Sigma : \Delta(C_{-S^l(m)}) \times \Lambda \mapsto \Delta(C_{S^l(m)})$  be a compact-valued correspondence that satisfies the upper and lower hemi-continuity.

$u_{S^{l(m)}}^\lambda : \Delta(C_{-S^{l(m)}}) \times \Lambda \times \Delta(C_{S^{l(m)}}) \rightarrow \mathbb{R}$  is a continuous function. Then, a function  $u : \Delta(C_{-S^{l(m)}}) \times \Lambda \rightarrow \mathbb{R}$  defined by

$$u(\sigma_{-S^{l(m)}}, \lambda) := \max_{\sigma_S \in \Sigma(\sigma_{-S^{l(m)}}, \lambda)} u_{S^{l(m)}}^\lambda((\sigma_{-S^{l(m)}}, \lambda), \sigma_{S^{l(m)}})$$

is a continuous function for any  $S^{l(m)}$ , by the Berge maximum theorem. Hence,  $F_{S^m}(\tilde{\pi}, \pi)$  is continuous in  $\lambda$ , and so is  $v_\lambda^\alpha(S^m)$ . Then,  $\varphi(v_\lambda^\alpha)$  is continuous in  $\lambda$ , since from the formula giving the Shapley value it is clearly that  $\varphi(v_\lambda^\alpha)$  is continuous in each  $v_\lambda^\alpha(S^m)$ .

(2) Under a coalition structure  $\tilde{P} = [N]$ , players maximize  $(u_i(\sigma_N))_{i \in N}$ , where  $u_i(\sigma_N) = \sum_{c_N \in C_N} \sigma_N(c_N) u_i(c_N)$ . Clearly  $v_\lambda^\alpha(\{i\}) \leq u_i(\bar{\sigma}_N)$  for any  $\lambda$ . Hence,  $v_\lambda^\alpha(\{i\}) \in V^\alpha(N)$  for any  $\lambda \in \Lambda$ , since  $(u_i(\bar{\sigma}_N))_{i \in N} \in V^\alpha(N)$ .

(3) Let  $\underline{u} = (\min_{\lambda \in \Lambda} v_\lambda^\alpha(\{i\}))_{i \in N}$ . Then,  $\underline{u} \in V^\alpha(N)$  by (2). By the definition of  $V^\alpha(N)$ , if  $u \in V^\alpha(N)$  and  $\underline{u} \leq u' \leq u$ , then  $u' \in V^\alpha(N)$ .

(4) Given a  $\lambda \in \Lambda$ , let  $y \in \mathbb{R}^n$  be a vector of payoffs such that  $\lambda * y = \varphi(v_\lambda^\alpha)$  and define  $\eta^\lambda \in \mathbb{R}_+$  by the condition that  $\eta(y - \underline{u}) + \underline{u} \in \text{int}V^\alpha(N)$  for  $\eta < \eta^\lambda$ ,  $\eta(y - \underline{u}) + \underline{u} \in \partial V^\alpha(N)$  for  $\eta = \eta^\lambda$ , and  $\eta(y - \underline{u}) + \underline{u} \notin V^\alpha(N)$  for  $\eta > \eta^\lambda$ . There is such an  $\eta^\lambda$  by (3). It is continuous in  $\lambda$  by the continuity of  $\varphi^\alpha(v_\lambda)$  shown in (1), and so  $\eta^\lambda y$  is continuous in  $\lambda$ .

(5) For  $\lambda \in \Lambda$  define a correspondence  $\Psi : \Lambda \mapsto \Lambda$  by

$$\Psi(\lambda) = \{\mu \in \Lambda \mid \mu \cdot y = \lambda \cdot y, \text{ and } \mu \cdot x \leq \mu \cdot \eta^\lambda y \ \forall x \in V^\alpha(N)\}.$$

$y = \phi^\alpha(G, \lambda)$  if and only if the corresponding  $\lambda$  is such that  $\lambda \in \Psi(\lambda)$ .

(6)  $\Psi$  is a mapping from  $\Lambda$  to  $\Lambda$ .  $\Lambda$  is a non-empty, compact convex subset in  $\mathbb{R}^n$ .  $\Psi$  is upper hemi-continuous, since  $\eta^\lambda y$  is continuous in  $\lambda$  by (4).  $\Psi$  is convex-valued, since  $V(N)$  is convex. Therefore, there exists a  $\lambda$  such that  $\lambda \in \Psi(\lambda)$ , by the Kakutani fixed point theorem. *Q.E.D.*

### The Consistency

Recall the 3-person game  $G^1$  with  $\lambda = (1, 1, \mu)$  in section 3. There is the dominant strategy equilibrium  $(b, L, q)$ , and so  $v_\lambda^\alpha(S) = 0$  for any coalition  $S \subseteq N$ . Since  $(0, 0, 0) \in V^\alpha(N)$ , the unique Shapley NTU value of  $G^1$  is

$$\phi_1^\alpha(G^1, \lambda) = 0, \quad \phi_2^\alpha(G^1, \lambda) = 0, \quad \phi_3^\alpha(G^1, \lambda) = 0.$$

The naive consistency is now confirmed in that game.

**Theorem 2**  $\phi^\alpha(G, \lambda)$  satisfies the dummy player consistency.

*Proof:* Let  $D \subset N$  be the set of dummy players in  $G$  and  $\phi^\alpha$ . Replacing all the payoffs  $u_j(\sigma)$  for every dummy player  $j$  with  $\phi_j^\alpha(G, \lambda) = 0$ , we have that  $u_{S^l(m)}^\lambda(\sigma) = \sum_{i \in S^l(m) \setminus D} \lambda_i u_i(\sigma) + \sum_{j \in D \cap S^l(m)} \lambda_j u_j(\sigma) = \sum_{i \in S^l(m) \setminus D} \lambda_i u_i(\sigma)$  for each subcoalition  $S^l(m)$ . However,  $\bar{\sigma}_{S^l(m)}$  is unchanged, since every  $j \in D$  has only one strategy and  $\bar{\sigma}$  is the dominant strategy equilibrium in  $G$ .

Since  $\bar{\sigma}$  is played under any  $\tilde{P}$ , we have  $v_\lambda^\alpha(\mathcal{P}_i^R \cup \{i\}) - v_\lambda^\alpha(\mathcal{P}_i^R) = \lambda_i u_i(\bar{\sigma})$  for any  $i \in N \setminus D$ . Since  $\phi_j^\alpha(G, \lambda) = \varphi_j(v_\lambda^\alpha) / \lambda_j = 0$  for any  $j \in D$ , we have  $\sum_{\mathcal{R}} v_\lambda^\alpha(\mathcal{P}_j^R \cup \{j\}) - v_\lambda^\alpha(\mathcal{P}_j^R) = 0$ , i.e.,  $\sum_{j \in D \cap S^l(m)} \lambda_j u_j(\bar{\sigma})$  is cancelled out before the replacements. Hence,  $\sum_{\mathcal{R}} v_\lambda^\alpha(\mathcal{P}_i^R \cup \{i\}) - v_\lambda^\alpha(\mathcal{P}_i^R)$  is the same both in  $G$  and in  $G'$ . Therefore,  $\phi^\alpha(G, \lambda) = \phi^\alpha(G', \lambda)$ . Q.E.D.

### The finiteness

Some strategic-form games have infinitely many Shapley NTU values, as shown in section 3. We show that the number of the Shapley NTU value is at most finite for almost every game in strategic form.

Define the *graph* of  $V^\alpha(N) = \text{con}\{(x_i)_{i \in N} | x_i \leq u_i(c), c \in \times_{i \in N} C_i\}$  by

$$\Gamma(V^\alpha(N)) = \{(x, \lambda) \in V^\alpha(N) \times \Lambda | x \cdot \lambda = v_\lambda^\alpha(N)\}.$$

For a finite collection  $(\Delta^i)_{i=1}^m$  of subsets in  $\Lambda$  such that  $\cup_{i=1}^m \Delta^i = \Lambda$ , define  $F^i = \{(x, \lambda) \in \mathbb{R}^n \times \Delta^i | x \cdot \lambda = v_\lambda^\alpha(N) \forall \lambda \in \Delta^i\}$ .  $\Pi_x(F^i) = \{x | (x, \lambda) \in F^i\}$  and  $\Pi_\lambda(F^i) = \{\lambda | (x, \lambda) \in F^i\}$  are the natural projections of  $F^i$ .

Since  $V^\alpha(N)$  is closed convex and bounded above and has only finitely many facets, we can take  $(\Delta^i)_{i=1}^m$  in such a way that  $\Gamma(V^\alpha(N)) \subset \cup_{i=1}^m F^i$  and  $V^\alpha(N) \cap \Pi_x(F^i) \subset \Pi_x(F^i)$  for  $i = 1, \dots, m$ .

For any  $S^m \subset N$ ,  $V^\alpha(S^m)$  is also closed convex and bounded above and has only finitely many vertices. At a vertex  $z = (\min_{\bar{\sigma}} u_i(\bar{\sigma}))_{i \in S^m}$  of  $V^\alpha(S^m)$  such that  $z \cdot \lambda = v_\lambda^\alpha(S^m)$ , we have  $z \cdot \lambda' = v_{\lambda'}^\alpha(S^m)$  with sufficiently small changes from  $\lambda$  to  $\lambda'$ . Hence, it is easy to show the following lemma, which is the key in the proof of Theorem 3.

**Lemma**  $v_\lambda^\alpha(S^m)$  is continuously differentiable with respect to  $\lambda \in \Pi_\lambda(F^i)$ ,  $i = 1, \dots, m$ , for any  $S^m \subset N$ .

We introduce a simplest perturbation into games to show the finiteness of the Shapley NTU values. Let  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$ . Given an original game  $G$ , the payoff function for each player  $i \neq n$  is replaced with  $u'_i(c) = u_i(c) + \epsilon_i$  for any  $c \in \times_{i \in N} C_i$ .  $u'_n(c) = u_n(c)$ . Define a perturbed game of an original game  $G$  by  $G^\epsilon = \{N, (C_i)_{i \in N}, (u'_i)_{i \in N}\}$ . Player  $n$  knows his own payoffs exactly but does not know “secret savings/debts” of the other players.

**Theorem 3** *Suppose that  $\lambda \in \Lambda$ . Then, in almost every game  $G$  in strategic form, the number of the Shapley NTU value of  $G$  is at most finite.*

*Proof:* See Appendix.

For each  $G^\epsilon$ , we can take a finite collection  $(\Delta^i)_{i=1}^m$  in  $\Lambda$  in such a way that  $\Gamma(V_\epsilon^\alpha(N)) \subset \cup_{i=1}^m F^i$  and  $V_\epsilon^\alpha(N) \cap \Pi_x(F^i) \subset \Pi_x(F^i)$  for  $i = 1, \dots, m$ . It is enough to show that each  $F_i$  has finitely many Shapley values of  $G^\epsilon$ .

We could show the differentiability of  $v_\lambda^\alpha$  by examining the NTU game  $V^\alpha$  corresponding to it via multiobjective optimizations. Since  $v_\lambda^H$  has no corresponding NTU game, we cannot confirm its differentiability in the same way. Hence, the finiteness of the Shapley values of  $G$  due to Harsanyi is unsolved. We should develop another device to overcome this difficulty.

## 6 Discussion

We can find a similar strategic foundation of cooperative games in the recent literature. Funaki and Yamato (1999) examined whether the core is empty or not in an economy with a common pool resource, formalizing their model as a game in partition function form where actions taken by coalitions satisfy the best response property in our sense. The major difference is that we introduce subpartitions in each coalition.

A simplified example is as follows. There are  $n$  workers jointly producing a commodity.  $x_i$  is the effort level that worker  $i$  exerts, and their production function is given by  $(x_N)^{1/2}$ , where  $x_N = \sum_i x_i$ . The price of the commodity is normalized to 1. The total revenue is shared proportionally to the effort level among all the  $n$  workers. The utility of each worker  $i$  is then given by

$$u_i(x_1, x_2, \dots, x_n) = \frac{x_i}{x_N} (x_N)^{1/2} - x_i.$$

Consider a coalition structure  $P = [S_1, \dots, S_k]$  (a “partition” in our definition). In each coalition  $S_j$ , workers cooperate to maximize their joint utility  $\sum_{i \in S_j} u_i$ . According to their Theorem 1, the equilibrium joint utility of coalition  $S_j$  is  $(2k - 1)/4k^3$  under any  $P = [S_1, \dots, S_k]$ .

However, if workers in some  $S_j$  did not cooperate as a single entity but gathered as two subcoalitions in  $S_j$  with the outsiders’ coalitions being unchanged, the equilibrium joint utility of  $S_j$  would be  $2(2(k + 1) - 1)/4(k + 1)^3$ , which is larger than  $(2k - 1)/4k^3$  if  $k \geq 3$ .

Hence, as shown with the example in section 4, subpartitioning in each coalition could start also in the model of Funaki and Yamato. They showed that the core is nonempty if each coalition has pessimistic anticipations of the coalition formation of the outsiders, but it may be empty otherwise.

The example in section 4 suggests the “steady” coalition structure where the subpartitioning process stops in every coalition. We also did not develop this notion in this paper, but with this approach Funaki and Yamato might have strengthened their main proposition in such a way that the result does not depend on whether the anticipations of outsiders’ coalition formation are optimistic or pessimistic. This point is left for our future research.

Needless to say, more criteria besides the naive consistency on representations of noncooperative game in coalitional form should be developed. The Shapley NTU values appear sometimes counterintuitive, since they are not necessarily in the core<sup>5</sup>. Hence, other solutions such as the NTU nucleolus defined in Nakayama (1983) also should be studied for that purpose.

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<sup>5</sup>Roth (1980) and Shafer (1980) argued the applicability of the Shapley NTU values: the unique Shapley NTU value of the following game is  $(1/3, 1/3, 1/3)$ .

$$\begin{aligned} V(\{i\}) &= \{(x_i) | x_i \leq 0\}, \text{ for } i = 1, 2, 3 \\ V(\{1, 2\}) &= \{(x_1, x_2) | x_1 \leq 1/2, x_2 \leq 1/2\} \\ V(\{1, 3\}) &= \{(x_1, x_3) | x_1 \leq p, x_3 \leq 1 - p, 0 < p < 1/2\} \\ V(\{2, 3\}) &= \{(x_2, x_3) | x_2 \leq p, x_3 \leq 1 - p, 0 < p < 1/2\} \\ V(\{1, 2, 3\}) &= \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq 1, x_1 \leq 1/2, x_2 \leq 1/2\} \end{aligned}$$

Roth considered that  $(1/2, 1/2, 0)$  is more reasonable, because  $1/2$  is the highest payoff available for player 1 and 2, and they can obtain it without player 3. Moreover,  $(1/2, 1/2, 0)$  is in the core. Aumann (1985b) pointed out that the different bargaining procedures result in different outcomes in this game, and showed that  $(1/2, 1/2, 0)$  is supported by a bargaining model. Montero and Okada (2001) is on this line.

Finally, our representation has some applications to the patent licensing problems in oligopolistic markets. See Watanabe and Tauman (2003) and Watanabe (2004).

## Appendix

### Proof of Theorem 3

**Definition 10** Let  $f : M^m \rightarrow N^n$  be a mapping from  $M^m$  to  $N^n$ , where  $M^m$  is an  $m$ -dimensional manifold and  $N^n$  is an  $n$ -dimensional manifold. We say that  $f$  is transversal to  $y \in N^n$  and denote by  $f \pitchfork y$  if  $Df(x)$  has rank  $n$  whenever  $f(x) = y$ , where  $x \in M^m$ .

**Transversality Theorem<sup>6</sup>** Let  $f : M^m \times K^k \rightarrow N^n$  be smooth and  $f \pitchfork y \in N^n$ . Then, for almost every  $z \in K^k$ ,  $f_z : M^m \rightarrow N^n$  defined by  $f_z(x) = f(x, z)$ , where  $x \in M^m$ , satisfies  $f_z \pitchfork y$ .

*Proof:* It is easily confirmed that  $u'_i(\sigma) = u_i(\sigma) + \epsilon_i \forall i \neq n, \forall \sigma \in \Delta(C_N)$ , and so the perturbed NTU game  $V_\epsilon^\alpha$  is characterized by

$$V_\epsilon^\alpha(S^m) = V^\alpha(S^m) + (\epsilon_i)_{i \in S^m \setminus \{n\}}, \quad \forall S^m \in N.$$

Given a  $\lambda \in \Lambda$ ,  $v_{\lambda, \epsilon}^\alpha(S^m) = v_\lambda^\alpha(S^m) + \sum_{i \in S^m \setminus \{n\}} \lambda_i \epsilon_i$  for any  $S^m \subset N$ . Hence,  $\Gamma(V_\epsilon^\alpha(N)) = \Gamma(V^\alpha(N))$  for any  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$ , and so, for each  $G^\epsilon$ , we can take the same finite collection  $(\Delta^i)_{i=1}^m$  in  $\Lambda$  in such a way that  $\Gamma(V_\epsilon^\alpha(N)) \subset \cup_{i=1}^m F^i$  and  $V_\epsilon^\alpha(N) \cap \Pi_x(F^i) \subset \Pi_x(F^i)$  for  $i = 1, \dots, m$ . It is enough to show that each  $F_i$  has finitely many Shapley values of  $G^\epsilon$ .

Note that  $(x, \lambda) \in F^i \subset \mathbb{R}^k$ , where  $k \leq n - 1$ . Define a mapping  $f^i : \mathbb{R}^k \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  by

$$f^i(x, \lambda, \epsilon) = (\varphi(v_{\lambda, \epsilon}^\alpha) - \lambda * x)_{-n}.$$

$f^i(x, \lambda, \epsilon) = 0$  if and only if  $x = \phi^\alpha(G^\epsilon, \lambda)$  and  $(x, \lambda) \in F^i$ . By Lemma,  $v_\lambda^\alpha(S^m)$  is continuously differentiable with respect to  $\lambda \in \Pi_\lambda(F^i)$  for any  $S^m \subset N$ , and so  $v_{\lambda, \epsilon}^\alpha(S^m)$  is continuously differentiable.  $\varphi(v_{\lambda, \epsilon}^\alpha)$  is a linear function of  $v_{\lambda, \epsilon}^\alpha$ , and thus  $f^i(x, \lambda, \epsilon)$  is smooth.

<sup>6</sup>See Guillemin and Pollack (1974) or other textbooks.

Now we have

$$\begin{aligned} \partial f^i(x, \lambda, \epsilon)/\partial \epsilon &= \partial(\varphi(v_{\lambda, \epsilon}^\alpha)_{-n})/\partial \epsilon \\ &= (1/n!) \begin{pmatrix} n(n-1)! & 0 & \cdots & 0 \\ 0 & n(n-1)! & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n(n-1)! \end{pmatrix} * \lambda_{-n} \\ &= I * \lambda_{-n}. \end{aligned}$$

$\partial f^i(x, \lambda, \epsilon)/\partial \epsilon$  has full rank for every  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$ . Accordingly,  $Df^i(x, \lambda, \epsilon)$  also has full rank. Hence, we have  $f^i(x, \lambda, \epsilon) \pitchfork \{0\}$ .

Given an  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$ , let  $f_\epsilon^i(x, \lambda) := f^i(x, \lambda, \epsilon)$ . Then,  $f_\epsilon^i(x, \lambda) \pitchfork \{0\}$  for almost every  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$  by the transversality theorem.

Since  $F^i$  and the image  $f_\epsilon^i(F^i)$  are of dimension  $k$ , the inverse mapping  $(f_\epsilon^i)^{-1}(0)$  is of dimension 0, which implies that  $(x, \lambda) \in F^i$  such that  $f_\epsilon^i(x, \lambda) = 0$  is a point. Hence, for almost every  $\epsilon \in \mathbb{R}^{N \setminus \{n\}}$ , every Shapley NTU value  $\phi^\alpha(G^\epsilon, \lambda)$  is locally isolated from the other values.

Since  $\Lambda$  is compact and  $\phi^\alpha(G, \lambda)$  is continuous in  $\lambda$  as shown in (1) in the proof of Theorem 1, the set of the Shapley NTU values is compact. Since every Shapley NTU value is locally isolated, the number of the Shapley NTU value of  $G$  must be at most finite for almost every game  $G$  in strategic form. *Q.E.D.*

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