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**Asymptotic Properties of the Shapley  
Value of Patent Licensing Games**

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The Shapley value of a patent licensing game:  
the asymptotic equivalence to non-cooperative results

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**Summary.** We consider the Shapley value of a patent licensing game played by firms and an outside patent holder of a cost-reducing innovation. The key problem is how to define the worth of each coalition when there is a subtle mixture of conflict and cooperation. Our main result is an asymptotic equivalence: with many small firms in the Cournot oligopolistic market, the Shapley value of the patent holder approximates the payoff he obtains in the non-cooperative games traditionally studied in the literature.

**Keywords and Phrases:** patent licensing, Shapley value, Cournot oligopoly coalition formation, core, bargaining set

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# 1 Introduction

Patent licensing problems in oligopolistic markets have been studied as non-cooperative games, e.g., licensing by means of fixed fees or royalty in Kamien and Tauman (1984, 1986) and by means of auction in Katz and Shapiro (1985, 1986). Concerning with the strategic interaction between the patent holder (innovator) and the firms in the industry, almost all the literature focuses on the optimal licensing mechanism for the innovator and its impact on the market structure and the dissemination of new technology<sup>1</sup>.

For instance, Kamien, Oren and Tauman (1992) compared the above three licensing mechanisms (fixed fees only, royalty only, and auction) for general demand functions, and Sen and Tauman (2002) recently considered the optimal combination of fixed fees and royalty for the innovator in the presence of linear demand.

In reality, however, mergers of firms and negotiations on how to split the total profit are observed in common practice: the innovator and the firms can sign binding contracts that describe the transfer of new technology from the innovator to the firms in detail, and this may be followed by mergers among firms into cartels before competition sets in. It is therefore reasonable to study the cooperative interaction between the innovator and the firms, while taking into account the ensuing future competition in the market. We introduce such a cooperative aspect to the patent licensing of a cost-reducing innovation in a Cournot industry<sup>2</sup>, and provide further insights into the non-cooperative results obtained in Kamien and Tauman (1986).

A key problem is how to define the worth of each coalition of players under a subtle mixture of conflict and cooperation, taking into account a sophisticated nature of events. Licensees can merge and form cartel  $S$  with enhanced oligopolistic power. But, non-licensees may react also by forming cartels. Then, the licensees in  $S$  might not merge into a single entity, but gather as smaller subcartels in  $S$  forming the headquarter-subsidiaries relationship. (This point is illustrated with an example in the next section.)

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<sup>1</sup>Kamien (1992) is a comprehensive survey.

<sup>2</sup>Kats and Tauman (1986) studied the asymptotic relationship between the Shapley value and core of a production economy where only a limited number of permitted firms have access to a better production technology, assuming that every firm is a “price taker”.

Suppose that the firms compete *à la* Cournot and that this competition uniquely determines equilibrium profits for every coalition structure. In the spirit of von-Neumann and Morgenstern's minimax, we define the worth  $v(S)$  of each coalition  $S$  as the highest equilibrium profits that  $S$  can guarantee.

Our main result is an asymptotic equivalence. With many small firms, the Shapley value of the innovator in the above cooperative game approximates his payoff in the non-cooperative games traditionally studied in the literature<sup>3</sup>. That shows the robustness of the private value of a patent for large markets. The result is the same whether the approach is a cooperative one or a non-cooperative one.

The outline of the paper is as follows. In Section 2, we motivate our work with an example. The general definition of the worth of each coalition and solution concepts are formally described in section 3. Theorems are shown in section 4. The classical derivation of the worth of a coalition is briefly reviewed in Discussions. We show that if we apply the classical derivation, the patent holder will accept zero payoff in a large industry. This is remarkably against our intuition. Some related literature is discussed in the last section.

## 2 An Example

Consider a Cournot competition by 7 firms, each producing  $q_i$  units of an identical commodity with the unit cost  $c$ . The market price  $p$  is determined by  $p = a - \sum q_i$ , where  $a(> c)$  is a positive constant. Suppose that 3 coalitions are formed, each consisting of 4 firms, 2 firms, and 1 firm. The objective of firms in the same coalition is to maximize the total profit of their coalition via a cartelization of its members, no matter how the other players form their own cartel structures.

If firms in each coalition  $S$  cooperated simply to maximize the total profits  $\sum_{i \in S} pq_i - cq_i$ , the equilibrium joint profit of each coalition would then be  $((a - c)/4)^2$ . That is the same amount as each firm obtains in the 3-person Cournot competition, because firms in each coalition operated as if they were a single entity.

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<sup>3</sup>Kamien and Tauman (1986) and Kamien, Oren and Tauman (1992).

Firms in each coalition can now jointly decide whether to merge into a single entity or to gather as some subcoalitions in the coalition. In the market, firms in the same subcoalition cooperate to maximize the total sum of individual profits in the subcoalition, while each subcoalition competes *à la* Cournot not only with the “opponent” subcoalitions outside  $S$  but also with the other “colleague” subcoalitions in  $S$ . Let  $n'$  be the number of subcoalition that supply the commodity to the market. The following is easily confirmed.

**Observation** *The equilibrium profit of each subcoalition is  $((a-c)/(n'+1))^2$ .*

The joint profit of each coalition is naturally defined as the total sum of profits of its subcoalitions. Coalition  $S^*$  of 4 firms is now in question. Suppose that the other coalitions remained as a single entity. It would then be best for  $S^*$  to have 3 subcoalitions, since it could receive the larger equilibrium joint profit  $3((a-c)/6)^2$ . In this case, the coalition of 2 firms would receive  $((a-c)/6)^2$ . But if it has 2 subcoalitions, it will receive a larger total profit of  $2((a-c)/7)^2$ . Anticipating such a counter coalition formation by the rival coalition, it is best for coalition  $S^*$  to have 4 subcoalitions and receive the equilibrium joint profit of  $4((a-c)/8)^2$ .

Coalition  $S^*$  can retain  $4((a-c)/8)^2$  by having 4 subcoalitions even in the worst anticipation that rival coalitions would form a coalition structure with 3 subcoalitions in total so that they would minimize the equilibrium joint profit of  $S^*$ . Hence,  $v(S^*) = 4((a-c)/8)^2$  is regarded as the “guaranteed” worth of coalition  $S^*$ .

In the next section, we introduce a patent holder into a Cournot industry. It is easy to compute the Shapley value of the patent holder and 7 firms exactly, but the expression would be quite complicated in the case of more firms. We hence approximate it for a large industry size.

The standard approximation is by replicating each firm in the market  $k$  times to generate  $k$  identical firms of the same type (and total of  $nk$  firms). Then, the limits are taken when  $k$  tends to infinity. This is done precisely in the next section.

### 3 The Model

#### A Cooperative Patent Licensing Game

Let  $N = \{1, 2, \dots, n\}$  be the set of  $n$  firms, where  $2 \leq n < \infty$ . Each firm  $i \in N$  produces  $q_i (\geq 0)$  units of an identical commodity with the same unit cost  $c (> 0)$  of production. The market price  $p$  of the commodity is determined by  $p = \max(0, a - \sum_{i \in N} q_i)$ , where  $a \in (c, \infty)$  is a constant.

An innovator outside the market has a patent of a new technology which reduces the unit cost of production from  $c$  to  $c - \epsilon$ , where  $0 < \epsilon < c$  and  $a - c - \epsilon > 0$  (non-drastic innovation<sup>4</sup>). The profit of firm  $i$  is  $u_i(q) = pq_i - C(q_i)$ .  $C(q_i) = (c - \epsilon)q_i$ , if  $i$  is a licensee, and  $C(q_i) = cq_i$ , otherwise. The patent holder has no production facilities, but shares some of the profits of licensees in return for licensing his innovation. The set of players in the patent licensing game is  $N' = \{0\} \cup N$ . Player 0 is the patent holder.

The game has two stages. (1) The patented innovation is licensed to firms, and payments are made to the patent holder. Some of the licensees may merge and form cartels. Non-licensees can also form cartels. (2) Under the cartel structure determined in the first stage, firms finally compete in the market *à la* Cournot, each of which maximizing the joint profit of the cartel it belongs to.

We describe later the more sophisticated coalition formation, and analyze this game in a cooperative way: how much could the patent holder gain through the negotiation with firms?

To approximate a large market, we replicate each one of the  $n$  firms  $k$  times and hence generate  $k$  identical firms from each firm. The patent holder is the only agent who is not being replicated. Let  $E_k$  be the Cournot oligopolistic market which is generated by this replication.

Denote by  $N_k$  the set of the  $nk$  firms in  $E_k$  and let  $N'_k = \{0\} \cup N_k$ .  $N$  is regarded as the set of types. Denote by  $N_k(i)$  the set of the  $k$  firms of type  $i \in N$ . All the firms in  $N_k$  are identical because they all have the same cost function prior to the licensing stage.

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<sup>4</sup>The drastic case, i.e.,  $a - c - \epsilon \leq 0$ , is trivial to analyze, since every firm without the new technology will stop the production in this linear Cournot model.

Hereafter, let us consider the game backwardly from the second stage, according to the traditional literature.

Let  $P_k = [S^0, S^1, \dots, S^M]$  denote a partition of  $N_k$ . Each nonempty element is called a *coalition* of firms.  $S^0$  consists of all the licensees. Each coalition  $S^m$  with  $m \neq 0$  consists of only non-licensees. For every  $m$ , let  $\tilde{S}^m = \{S^{1(m)}, \dots, S^{l(m)}, \dots, S^{L(m)}\}$  be a subpartition of  $S^m$  into nonempty subsets. Each  $S^{l(m)} \subset S^m$  is a *subcoalition* of firms. Denote by  $\tilde{P}_k = [\tilde{S}^0, \tilde{S}^1, \dots, \tilde{S}^M]$  a *coalition structure* of firms<sup>5</sup>.

Given a coalition structure, each subcoalition  $S^{l(m)}$  competes *à la Cournot* with all the other subcoalitions including “colleague” subcoalitions in  $\tilde{S}^m$ , in the second stage. Assume that  $q_j$  is a continuous variable for every firm  $j$  to ensure the existence of  $\bar{q}$  in pure strategies.

**Definition 1** Given a coalition structure  $\tilde{P}_k$  in the replicated market  $E_k$ , we say that a strategy profile  $\bar{q} = (\bar{q}_j)_{j \in N_k}$  satisfies the **best response property** under  $\tilde{P}_k$  if for every subcoalition  $S^{l(m)}$  in every coalition  $\tilde{S}^m$  there is no  $q_{S^{l(m)}} = (q_j)_{j \in S^{l(m)}}$  such that  $u_{S^{l(m)}}(q_{S^{l(m)}}, \bar{q}_{-S^{l(m)}}) > u_{S^{l(m)}}(\bar{q})$ . Denote by  $\beta(\tilde{P}_k)$  the set of such strategies.

Note that the cost functions are linear and firms in the same subcoalition have identical cost functions. Hence, operating several firms in  $S^{l(m)}$  who act cooperatively (namely, make their production decision jointly) is equivalent to operating a single firm. Thus, without loss of generality, it can be assumed that  $S^{l(m)}$  is a single firm  $j$  having the cost function  $C(q_j) = cq_j$  if  $m \neq 0$  and  $C(q_j) = (c - \epsilon)q_j$  if  $m = 0$ .

Let  $\pi_k(S^m)$  denote the set of all the subpartitions of  $S^m$ . Denote by  $\tilde{\pi}_k(N_k \setminus S^m)$  the sets of all the coalition structures defined over  $N_k \setminus S^m$ . A coalition structure  $\tilde{P}_k = [\tilde{S}^0, \tilde{S}^1, \dots, \tilde{S}^M]$  is rewritten from the viewpoint of  $S^m$  as  $\tilde{P}_k = [\tilde{\pi}_k, \pi_k]$ , where  $\tilde{\pi}_k \in \tilde{\pi}_k(N_k \setminus S^m)$  and  $\pi_k \in \pi_k(S^m)$ .

Given the outcomes determined in the competition stage, coalition  $S^m$  would choose its subpartition  $\pi_k$  anticipating the most antagonistic coalition structure  $\tilde{\pi}_k$  by the other players outside it, in the spirit of von Neumann and Morgenstern’s minimax.

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<sup>5</sup>Our definition is slightly different from the usual one. A partition  $P_k$  of  $N_k$  is usually called a coalition structure. We introduced subcoalitions in each coalition.

Denote the total equilibrium profit of each coalition  $S^m$  of firms under  $\tilde{P}_k = [\tilde{\pi}_k, \pi_k]$  by  $\sum_{S^{l(m)} \in \pi_k} u_{S^{l(m)}}(\bar{q}) = \sum_{S^{l(m)} \in \pi_k} \sum_{j \in S^{l(m)}} u_j(\bar{q})$ . Now, we are at the beginning of the first stage.

**Definition 2** Let  $v_k$  be a v-function that assigns to each  $S \subset N'_k$  a real number  $v_k(S)$ . The **guaranteed worth**  $v_k(S)$  of group  $S$  is defined as

$$\begin{aligned} v_k(S^0 \cup \{0\}) &= \min_{\tilde{\pi}_k \in \tilde{\pi}_k(N_k \setminus S^0)} \max_{\pi_k \in \pi_k(S^0)} f_{S^0}(\tilde{\pi}_k, \pi_k) \\ v_k(S^m) &= \min_{\tilde{\pi}_k \in \tilde{\pi}_k(S^0 = N_k \setminus S^m)} \max_{\pi_k \in \pi_k(S^m)} f_{S^m}(\tilde{\pi}_k, \pi_k), \quad m \neq 0 \end{aligned}$$

where  $f_{S^m}(\tilde{\pi}_k, \pi_k) = \min_{\bar{q} \in \beta(\tilde{P}_k)} \sum_{S^{l(m)} \in \pi_k} u_{S^{l(m)}}(\bar{q})$  with  $\tilde{P}_k = [\tilde{\pi}_k, \pi_k]$  given, and  $v_k(\{0\}) = v_k(\phi) = 0$ . The cooperative patent licensing game in the replicated market  $E_k$  is defined as  $(N'_k, v_k)$ .

$S^0 = N_k \setminus S^m$  is the worst case for  $S^m$  with  $m \neq 0$ , since all the other firms are licensed when some firms form a coalition  $S^m$  without the license.

## Solution Concepts

We here describe the definition of the Shapley value of the  $k$ -replicated game  $(N'_k, v_k)$ . The definitions of the core and bargaining set are given later in the following section.

Let  $\mathcal{R} : j_0, j_1, j_2, \dots, j_{nk}$  be an ordering of  $nk + 1$  players.  $j' \mathcal{R} j$  means that  $j'$  precedes  $j$  in the ordering  $\mathcal{R}$ . Define the set of players who precede player  $j \in N'_k$  in  $\mathcal{R}$  as  $\mathcal{P}_j^{\mathcal{R}} = \{j' \in N'_k | j' \mathcal{R} j\}$ .

**Definition 3** The **Shapley value**  $\text{Sh}(v_k) = (\text{Sh}_j(v_k))_{j \in N'_k}$  of a replicated game  $(N'_k, v_k)$  is defined as

$$\text{Sh}_j(v_k) = \frac{1}{(nk + 1)!} \sum_{\mathcal{R}} v_k(\mathcal{P}_j^{\mathcal{R}} \cup \{j\}) - v_k(\mathcal{P}_j^{\mathcal{R}}).$$

The Shapley value of type  $i$  is hence defined as  $\text{Sh}_i(v_k) = \sum_{j \in N_k(i)} \text{Sh}_j(v_k)$ .

Denote by  $s = |S^0|$  the number of licensees. Let  $K = (a - c)/\epsilon (> 1)$ . In general,  $K$  is not necessarily an integer. Only for a simpler description, however, we will hereafter treat it as an integer, according to the traditional literature<sup>6</sup>.

<sup>6</sup>See Kamien (1992), etc. Denote by  $\lfloor K \rfloor$  the integer  $z$  such that  $K - 1 \leq z < K$ . The precise analysis is done with  $\lfloor K \rfloor$ .

## 4 Asymptotic Results

**Result 1** Let  $s = |S^0| \in [1, nk - 1]$ , and let  $m \neq 0$ .

$$\begin{aligned}
v_k(S^0 \cup \{0\}) &= s \left( \frac{a-c+(1+nk-s)\epsilon}{1+nk} \right)^2 && \text{if } s < \min(K, 2 + nk - s) \\
&K \left( \frac{a-c+\epsilon}{1+K} \right)^2 && \text{if } K \leq \min(s, 1 + nk - s) \\
(1 + nk - s) \left( \frac{a-c+(1+nk-s)\epsilon}{2(1+nk-s)} \right)^2 &&& \text{if } 1 + nk - s < \min(s, K) \\
\left( \frac{a-c+\epsilon}{2} \right)^2 &&& \text{if } S^0 = N'_k \\
\\
v_k(S^m) &= 0 && \text{if } |S^m| \leq nk - K \\
|S^m| \left( \frac{a-c-(nk-|S^m|)\epsilon}{1+nk} \right)^2 &&& \text{if } nk - K < |S^m| \leq \frac{1+nk}{2} \\
(1 + nk - |S^m|) \left( \frac{a-c-(nk-|S^m|)\epsilon}{2(1+nk-|S^m|)} \right)^2 &&& \text{if } \max\left(\frac{1+nk}{2}, nk - K\right) < |S^m| \\
\left( \frac{a-c}{2} \right)^2 &&& \text{if } S^m = N_k
\end{aligned}$$

$v_k(S^m)$  depends on the magnitude  $\epsilon$  of the cost reduction, the size  $|S^m|$  of the coalition, and the market size  $nk$ . The intuition is as follows.

(1) As shown in the previous example, it is optimal for any coalition to have exactly one more subcoalitions than the subcoalitions outside it, if possible. If not, the coalition should have as many subcoalitions as possible.

(2) With the technological heterogeneity due to licensing, there is an additional policy: if  $K \leq \min(s, 1 + nk - s)$ , then it is best for coalition  $S^0$  to induce every non-licensee stop the production by having  $K$  subcoalitions.

Similarly, if  $|S^m| \leq nk - K$ , then the other firms outside  $S^m$  can form a coalition  $S^0 = N_k \setminus S^m$  together with the patent holder, have  $K$  subcoalitions, and induce every non-licensee in  $S^m$  stop the production.

(3) Finally, it is worse for any coalition to have more rival subcoalitions.

Result 1 suggests that if  $K \leq s \leq 1 + nk - K$ , then it is optimal for  $S^0$  to induce every non-licensee stop the production by having  $K$  subcoalitions. The equilibrium joint profit of  $S^0$  is then  $K \left( \frac{a-c+\epsilon}{1+K} \right)^2 = \epsilon(a - c)$ . The non-licensee obtains the zero profit. Hence, we can see the following.

**Proposition 1** *When the patent holder can choose the number  $s$  of licensees in such a way that  $K \leq s \leq nk/2$ , he can take  $\epsilon(a - c)$  as the lower bound of his revenue, if he can replace any licensees insisting positive payoffs with non-licensees.*

## The Shapley Value

Fix an ordering  $\mathcal{R}(s+1)$  of all the players where the patent holder is at the  $(s+1)$ -st place. Let  $\mathcal{P}_0^{\mathcal{R}(s+1)} = \{j \in N'_k | j \mathcal{R}(s+1) 0\}$  denote the set of firms  $j$ 's who precede the patent holder 0 in  $\mathcal{R}(s+1)$ . There are  $nk!$  orderings with the same marginal contribution  $v_k(\mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}) - v_k(\mathcal{P}_0^{\mathcal{R}(s+1)})$  of 0 to coalition  $S^0 \cup \{0\} = \mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}$ , by symmetry of firms. Hence,

$$\text{Sh}_0(v_k) = \frac{1}{nk+1} \sum_{s=1, \dots, nk+1} v_k(\mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}) - v_k(\mathcal{P}_0^{\mathcal{R}(s+1)}).$$

For firm  $j$  of type  $i$ ,  $\text{Sh}_0(v_k) + nk\text{Sh}_j(v_k) = v_k(N'_k)$  and  $\text{Sh}_i(v_k) = \sum_{j \in N_k(i)} \text{Sh}_j(v_k) = k\text{Sh}_j(v_k)$  by efficiency and symmetry<sup>7</sup>. Hence,

$$\text{Sh}_i(v_k) = (1/n)\{v_k(N'_k) - \text{Sh}_0(v_k)\} \text{ for all type } i \in N.$$

Consider Result 1 in the case of  $nk-1 > 2K$ . Then, we have

$$v_k(\mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}) = \begin{cases} s \left( \frac{a-c+(1+nk-s)\epsilon}{1+nk} \right)^2 & \text{if } 1 \leq s \leq K-1 \\ K \left( \frac{a-c+\epsilon}{1+K} \right)^2 & \text{if } K \leq s \leq 1+nk-K \\ (1+nk-s) \left( \frac{a-c+(1+nk-s)\epsilon}{2(1+nk-s)} \right)^2 & \text{if } 2+nk-K \leq s \\ \left( \frac{a-c+\epsilon}{2} \right)^2 & \text{if } s = nk \end{cases}$$

$$v_k(\mathcal{P}_0^{\mathcal{R}(s+1)}) = \begin{cases} 0 & \text{if } s \leq nk-K \\ (1+nk-s) \left( \frac{a-c-(nk-s)\epsilon}{2(1+nk-s)} \right)^2 & \text{if } 1+nk-K \leq s \\ \left( \frac{a-c}{2} \right)^2 & \text{if } s = nk. \end{cases}$$

**Theorem 1**  $\lim_{k \rightarrow \infty} \text{Sh}_0(v_k) = \epsilon(a-c)$  and  $\lim_{k \rightarrow \infty} \text{Sh}_i(v_k) = \frac{1}{n} \left( \frac{a-c-\epsilon}{2} \right)^2$  for all type  $i \in N$ .

*Proof:* If  $s \in [nk-K, nk]$ , then  $0 \leq nk-s \leq K < \infty$ . Hence, as  $k \rightarrow \infty$ ,

$$\begin{aligned} \text{Sh}_0(v_k) &= \frac{1}{nk+1} \sum_{s=1}^{K-1} s \left( \frac{a-c+(1+nk-s)\epsilon}{1+nk} \right)^2 + \frac{1}{nk+1} \sum_{s=K}^{nk-K} K \left( \frac{a-c+\epsilon}{1+K} \right)^2 \\ &\quad + \frac{1}{nk+1} \left\{ K \left( \frac{a-c+\epsilon}{1+K} \right)^2 - \frac{(a-c-(nk-s)\epsilon)^2}{4(1+nk-s)} \right\} \\ &\quad + \frac{1}{nk+1} \sum_{r=2+nk-K}^{nk-1} \left\{ \left\{ \frac{a-c+(1+nk-s)\epsilon}{4(1+nk-s)} \right\}^2 - \left\{ \frac{a-c-(nk-s)\epsilon}{4(1+nk-s)} \right\}^2 \right\} \\ &\quad + \frac{1}{nk+1} \left\{ \left( \frac{a-c+\epsilon}{2} \right)^2 - \left( \frac{a-c}{2} \right)^2 \right\} \rightarrow K \left( \frac{a-c+\epsilon}{1+K} \right)^2 = \epsilon(a-c), \end{aligned}$$

<sup>7</sup>Firms in different subcoalitions have different total profits to maximize. However, they can roll dice to determine which subcoalition they are to be assigned to. Then, we can consider them as symmetric, since they are identical prior to licensing,

since  $(a - c + (1 + nk - s)\epsilon)^2/4(1 + nk - s) - (a - c - (nk - s)\epsilon)^2/4(1 + nk - s) < \infty$ . Then,  $\lim_{k \rightarrow \infty} \text{Sh}_i(v_k) = (1/n)\{((a - c + \epsilon)/2)^2 - \lim_{k \rightarrow \infty} \text{Sh}_0(v_k)\} = (1/n)((a - c - \epsilon)/2)^2$ . *Q.E.D.*

As noted, it is optimal for coalition  $S^0$  to have  $K$  subcoalitions, which is the number at which the equilibrium price just equals the marginal cost  $c$  of production of non-licensees. Then, every non-licensee stops its production and obtains the zero profit. For a large industry size, the patent holder can take all the profits of subcoalitions consisting of licensees, since he can replace any licensee objecting to the zero payoff with an outside non-licensee.

### **the Effect of the Anti-Trust Law**

Even if any merger and cooperation are prohibited in the market by law, the members in  $S^0$  can still cooperate to share the license in the most efficient way: Let  $\Pi_L(t)$  and  $\Pi_N(t)$  denote each licensee's profit and each non-licensee's in the market when  $t$  firms are licensed, respectively.

$$\begin{aligned}\hat{v}_k(S^0 \cup \{0\}) &= \max\{t\Pi_L(t) + (s - t)\Pi_N(t) | 0 \leq t \leq s\} \\ \hat{v}_k(S^m) &= \max\{0, |S^m| \cdot \Pi_N(nk - |S^m|)\},\end{aligned}$$

For  $s = |S^0| \geq K$ ,  $\hat{v}_k(S^0 \cup \{0\}) = K((a - c + \epsilon)/(1 + K))^2$  and  $\hat{v}_k(S^m) = 0$  for any  $S^m$ . It is easy to show the following.

**Proposition 2** *Suppose that any cooperation is prohibited in the market. Then,  $\lim_{k \rightarrow \infty} \text{Sh}_0(\hat{v}_k) = \epsilon(a - c)$  and  $\lim_{k \rightarrow \infty} \text{Sh}_i(\hat{v}_k) = 0$  for all type  $i \in N$ .*

Kamien and Tauman (1986) showed that if there are more than  $K$  firms in the market, the patent holder's revenue is then  $\epsilon(a - c)$  and every firm obtains nothing, when the new technology is non-cooperatively licensed.

In reality, the anti-trust law does not necessarily prohibit all types of cooperation and mergers in the market. It is easy to infer the following from Theorem 1 and Proposition 2.

**Proposition 3** *For a large industry size, the outside patent holder of a cost-reducing technology in the Cournot oligopolistic market obtains the same revenue by implementing the Shapley value as he obtains by means of non-cooperative licensing schemes, under any permissible coalition structures.*

## The Emptiness of the Core

Denote by  $w = (w_0, w_1, \dots, w_{nk}) \in \mathbb{R}^{nk+1}$  a payoff vector, where  $w_0$  is the patent holder's payoff and  $w_j$  is firm  $j$ 's. The set of imputations of  $v_k$  is  $I(v_k) = \{w \in \mathbb{R}^{nk+1} : \sum_{j \in N'_k} w_j = v_k(N'_k) \text{ and } w_j \geq v_k(\{j\}) \forall j \in N'_k\}$ .

**Definition 4** The **core** of  $(N'_k, v_k)$  is defined as

$$\text{Core}(v_k) = \{w \in I(v_k) : \sum_{j \in S} w_j \geq v_k(S) \text{ for all } S \subset N'_k\}$$

For a payoff vector  $w \in \text{Core}(v_k)$ , the core payoff  $x_i$  of type  $i \in N$  is defined as  $x_i = \sum_{j \in N_k(i)} w_j$ .

The next theorem is on the limit core. The core could be nonempty for  $k$  such that  $nk < \infty$ , depending on the magnitude  $\epsilon$  of the cost reduction<sup>8</sup>.

**Theorem 2**  $\lim_{k \rightarrow \infty} \text{Core}(v_k) = \emptyset$

*Proof*: See Appendix.

A group  $S^0 \cup \{0\}$  of the patent holder and licensees can guarantee to itself the joint profit of  $\epsilon(a - c)$ , by producing  $a - c$  which induces every non-licensee firm to stop the production. On the other hand, non-licensees can together claim the total profit of  $((a - c)/2)^2$  by forming a coalition  $N_k$  excluding the patent holder. Those two claims collide with each other over how to divide the total profit  $((a - c + \epsilon)/2)^2$  of the grand coalition  $N'_k$ .

Theorem 2 might suggest that in the large industry the patent holder license his innovation by means of fixed fees (or royalty) only than negotiate with all the firms over how to divide the total fruits of the cooperation by all, since there should be some group(s) that breaks off the negotiation.

The next question should then be asked: can a bargaining solution that is weaker than the core allow implementing the Shapley value? We next consider the bargaining set for that purpose.

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<sup>8</sup>Watanabe and Muto (2004) showed that for any permissible coalition structure, the core with coalition structure is empty, assuming that "any" cooperations among firms are not allowed in the linear Cournot oligopolistic market.

## The Exclusion from the Bargaining Set

For a coalition  $S$ , we say that a payoff vector  $x \in \mathbb{R}^{|S|}$  is  $S$ -feasible if  $\sum_{j \in S} x_j \leq v_k(S)$ . A pair  $(S, x)$  of a coalition  $S$  and an  $S$ -feasible payoff vector is called an *objection* by  $j$  against  $j'$  to  $w$  if  $j \in S$  but  $j' \notin S$  and  $x_j > w_j$  for all  $j \in S$ .

A pair  $(T, y)$  of a coalition  $T$  and a  $T$ -feasible payoff vector  $y$  is called a *counterobjection* by  $j'$  to the objection  $(S, x)$  of  $j$  if  $j' \in T$  but  $j \notin T$ , and  $y_{j'} \geq w_{j'}$  for all  $j' \in T \setminus S$  and  $y_{j'} \geq x_{j'}$  for all  $j' \in T \cap S$ .

**Definition 5** The **bargaining set** of a game  $(N'_k, v_k)$  is the set of all the imputation  $w \in I(v_\lambda)$  such that for every objection  $(S, x)$  of any player  $j$  against any other player  $j'$  to  $w$  there is a counterobjection  $(T, y)$  by  $j'$  to  $(S, x)$ . Denote by  $M(v_k)$  the bargaining set<sup>9</sup>.

The limit bargaining set is defined as  $\lim_{k \rightarrow \infty} M(v_k)$ . It is well known that  $M(v_k) \neq \emptyset$  if  $k < \infty$ , whereas  $\lim_{k \rightarrow \infty} M(v_k)$  may be empty. But, it is easy to show that an imputation  $w$  such that  $w_0 = \epsilon(a-c) - (2(a-c)\epsilon - \epsilon^2)/8$  and  $w_i = (1/n)\{(a-c-\epsilon)/2\}^2 + (2(a-c)\epsilon - \epsilon^2)/8\}$  for all type  $i \in N$  is in the limit bargaining set, and so the limit bargaining set is nonempty. We will omit the proof<sup>10</sup>.

**Theorem 3**  $\lim_{k \rightarrow \infty} \text{Sh}(v_k) \not\subset \lim_{k \rightarrow \infty} M(v_k) \neq \emptyset$ .

*Proof*: See Appendix.

$\text{Core}(v_k) \subset M(v_k)$  if  $\text{Core}(v_k) \neq \emptyset$ , since an imputation  $w$  is in the  $\text{Core}(v_k)$  if and only if no player has an objection against any other players to  $w$ . Theorem 3 implies that it is difficult to implement the Shapley value even through negotiations with a more strict requirement on objections than the core. In the proof,  $v_k(N'_k) - v_k(N_k) < \epsilon(a-c)$  is used as a crucial condition. It is well known that the Shapley value is not necessarily in the (nonempty) core, but in our model it is outside even the Bargaining set.

<sup>9</sup>There are many variations of the original definition of the bargaining set by Aumann and Maschler (1964). Watanabe and Muto (2004) derived the lower and upper bounds of the bargaining set with coalition structure.

<sup>10</sup>It is not difficult to show that the imputation  $w$  is asymptotically equivalent to the nucleolus.

## 5 Discussions

According to von Neumann and Morgenstern (1944), define the worth of coalition  $S^m \subset N'_k$  by

$$\begin{aligned} v'_k(S^0 \cup \{0\}) &= \min_{q_{-S^0}} \max_{q_{S^0}} \sum_{j \in S^0} u_j(q_{S^0}, q_{-S^0}), \\ v'_k(S^m) &= \min_{q_{-S^m}} \max_{q_{S^m}} \sum_{j \in S^m} u_j(q_{S^m}, q_{-S^m}), \end{aligned}$$

where  $u_j(q_{S^m}, q_{-S^m}) = \max(a - Q_{S^m} - Q_{-S^m}, 0)q_j - C(q_j)$ ,  $C(q_j) = (c - \epsilon)q_j$  if  $j \in S^0$  and  $C(q_j) = cq_j$  otherwise. The  $v$ -function  $v'_k$  is easily computed:  $v'_k(S^m) = 0$  for any  $S^m \neq S^0 \cup \{0\}$ ,  $N_k$ , when  $Q_{S^0} = a - c$ , while  $v'(S^0 \cup \{0\}) = 0$  for  $S^0 \cup \{0\} \neq N'_k$  when  $Q_{-S^0} = a - c + \epsilon$ .  $v'_k(N_k) = ((a - c)/2)^2$  and  $v'_k(N'_k) = ((a - c + \epsilon)/2)^2$ . Hence,

$$\begin{aligned} \lim_{k \rightarrow \infty} \text{Sh}_0(v'_k) &= \lim_{k \rightarrow \infty} \frac{1}{nk+1} \left\{ \left( \frac{a-c+\epsilon}{2} \right)^2 - \left( \frac{a-c}{2} \right)^2 \right\} = 0 \\ \lim_{k \rightarrow \infty} \text{Sh}_i(v'_k) &= (1/n) \left\{ \left( \frac{a-c+\epsilon}{2} \right)^2 - \lim_{k \rightarrow \infty} \text{Sh}_0(v'_k) \right\} = (1/n) \left( \frac{a-c+\epsilon}{2} \right)^2, \end{aligned}$$

for every type  $i \in N$ .

Moreover, we have that  $\lim_{k \rightarrow \infty} \text{Sh}_0(v'_k) \in \lim_{k \rightarrow \infty} \text{Core}(v'_k) \neq \emptyset$ , since a payoff vector  $x \in R^{n+1}$  in the limit core with symmetric payoffs to firms is characterized as

$$\begin{aligned} x_0 + nx_i &= \left( \frac{a-c+\epsilon}{2} \right)^2 \\ \left( \frac{a-c+\epsilon}{2} \right)^2 - \left( \frac{a-c}{2} \right)^2 &\geq x_0 \geq 0 \\ (1/n) \left( \frac{a-c+\epsilon}{2} \right)^2 &\geq x_i \geq 0, i \in N. \end{aligned}$$

for every type  $i \in N$ .

Since  $v'_k(S^m)$  is derived with the extremely pessimistic anticipation that the rival firms jointly minimize the payoffs of players in coalition  $S^m$ , the impact of the new technology is not so deserved as the one in  $v_k$ . Hence, we have  $\lim_{k \rightarrow \infty} \text{Sh}_0(v_k) > \lim_{k \rightarrow \infty} \text{Sh}_0(v'_k)$  and the set of negotiable payoffs are so enlarged that the limit core contains the asymptotic Shapley value.

The above result implies that even if the patent holder licenses the new technology to all the firms, he will accept zero payoff as the negotiation outcome for a large industry size. This is remarkably against our intuition. The classical derivation is less reasonable than ours at least in the patent licensing games.

We might have studied an alternative cooperative approach as well. Simplifying our  $v$ -function with some additional assumptions, Watanabe and Muto (2004) studied a licensing game with coalition structures. Their model is quite less specified: the market can be either the Cournot or the Bertrand oligopoly for homogeneous or differentiated commodities, the innovation can be a quality-improving technology as well as a drastic or non-drastic cost-reducing one, and the “spillover” of the patented innovation to non-licensees is also included in their model.

In such a generalized framework of licensing games traditionally studied, they considered (1) how many potential licensees the patent holder should invite to the negotiation on licensing and (2) how much profit sharing he can gain through negotiations with them. Their main result is that the bargaining set with coalition structure is determined as a singleton, if the number of licensees optimal for the patent holder is larger than that of non-licensees.

Another cooperative approach can be provided. In the Bertrand competition with differentiated commodities, it might be possible that a patent holder licenses the new cost-reducing innovation to a limited number of firms through negotiations with them on (1) how to cooperate in opposition to (collusive) pricing by the non-licensee(s) and (2) how to split the total sum of profits realized by licensees.

To formalize such a situation, we must consider coalition structures that are formed endogenously in the process of writing agreements: players are partitioned into some coalitions, each of which has a distribution rule on how to divide the joint profit of the coalition, and the action of each firm in each coalition should be compatible with the firm’s payoff maximization. Given a coalition structure with distribution rules, if every firm in every coalition “individually” agrees upon the action required to take in the coalition, the list of such actions is regarded as a “binding agreement”.

Watanabe (2004) investigated such an aspect of the cooperative patent licensing problem, extending a duopoly studied by Muto (1993) to a triopoly. The main result is that it is most profitable for the patent holder to license its new technology to two firms if the commodities are not so substitutive, or to only one firm if they are quite substitutive.

## Appendix

### Proof of Result 1: the v-function

We begin with the strategy profiles with the best response property. As noted, it can be assumed that  $S^{l(m)}$  is a single firm  $j$  having the cost function  $C(q_j) = cq_j$  if  $m \neq 0$  and  $C(q_j) = (c - \epsilon)q_j$  if  $m = 0$ , and  $K = (a - c)/\epsilon$  is treated as an integer.

Given a coalition structure  $\tilde{P}_k$ ,  $M$  is the number of coalitions of non-licensees in  $\tilde{P}_k$  and  $L(t)$  is the number of subcoalitions in coalition  $S^t$ . Let  $m' = \sum_{t \neq 0} L(t)$  and  $m'' = \sum_{t \neq 0, m} L(t)$ .  $L(0)$  is the number of subcoalitions of licensees and  $m'$  is the number of subcoalitions of non-licensees, respectively. Let  $Q_{S^{l(m)}} = \sum_{j \in S^{l(m)}} q_j$ . It is easy to show the following.

Given a coalition structure  $\tilde{P}_k$ , the corresponding strategy profile in  $\beta(\tilde{P}_k)$  is uniquely determined as below.

If  $1 \leq L(0) < K$  and  $M \geq 1$ , then

$$\bar{Q}_{S^{l(0)}} = \frac{a - c + (1 + m')\epsilon}{1 + L(0) + m'} \quad \forall l(0) \quad (1)$$

$$\bar{Q}_{S^{l(m)}} = \frac{a - c - L(0)\epsilon}{1 + L(0) + m'' + L(m)} \quad \forall l(m), \forall m. \quad (2)$$

If  $K \leq L(0)$  and  $M \geq 1$ , then

$$\bar{Q}_{S^{l(0)}} = \frac{a - c + \epsilon}{1 + L(0)} \quad \forall l(0) \quad (3)$$

$$\bar{Q}_{S^{l(m)}} = 0 \quad \forall l(m), \forall m. \quad (4)$$

If  $M = 0$ , then

$$\bar{Q}_{S^{l(0)}} = \frac{a - c + \epsilon}{1 + L(0)} \quad \forall l(0). \quad (5)$$

If  $L(0) = 0$ , then

$$\bar{Q}_{S^{l(m)}} = \left(\frac{a - c}{1 + m'}\right)^2 \quad \forall l(m), \forall m = 1, \dots, M. \quad (6)$$

(I) Let us derive  $v_k(S^0 \cup \{0\})$ . For  $S^0 \cup \{0\} \neq N'_k$ , it suffices to show the case of  $L(0) \leq K$ , since  $L(0)((a-c+\epsilon)/(1+L(0)))^2$  is decreasing in  $L(0)$  if  $K \leq L(0)$ . When  $L(0) < K$ , the equilibrium joint profit of  $S^0$  with its  $L(0)$  subcoalitions is  $E(L(0)) = L(0)((a-c+(1+m')\epsilon)/(1+L(0)+m'))^2$  by (1), and so

$$v_k(S^0 \cup \{0\}) = \min_{1 \leq m' \leq nk-s} \max_{1 \leq L(0) \leq s} E(L(0)).$$

$E(L(0))$  increasing in  $L(0)$  until  $1+m'$ , and decreasing in  $m'$ . Hence, it is then optimal for  $S^0$  to have  $L(0) = 1+m'$  subcoalitions. if  $L(0) \leq s$  and  $L(0) < K$  In summary,

$$v_k(S^0 \cup \{0\}) = \begin{cases} (1+nk-s)\left(\frac{a-c+(1+nk-s)\epsilon}{2(1+nk-s)}\right)^2 & \text{if } 1+nk-s < \min(s, K) \\ s\left(\frac{a-c+(1+nk-s)\epsilon}{1+nk}\right)^2 & \text{if } s < \min(K, 2+nk-s) \\ K\left(\frac{a-c+\epsilon}{1+K}\right)^2 & \text{if } K \leq \min(s, 1+nk-s). \end{cases}$$

When  $S^0 \cup \{0\} = N'_k$ ,  $v_k(S^0)$  is rewritten with (5) as

$$v_k(S^0 \cup \{0\}) = \max_{1 \leq L(0) \leq nk} L(0)\left(\frac{a-c+\epsilon}{1+L(0)}\right)^2 = \left(\frac{a-c+\epsilon}{2}\right)^2.$$

(II) Let us derive  $v_k(S^m)$  with  $m \neq 0$ . Suppose  $K \leq nk - |S^m|$ . Then, all the other firms can form a coalition  $S^0$  together with the patent holder and have  $K$  subcoalitions in order to have every non-licensee stop the production by (4). Accordingly,  $v_k(S^m) = 0$  if  $K \leq nk - |S^m|$ .

Consider the case of  $K > nk - |S^m|$ .  $v_k(S^m)$  is rewritten with (2) as

$$v_k(S^m) = \min_{0 \leq m'' \leq nk-L(0)-|S^m|} \max_{1 \leq L(0) \leq nk-|S^m|} D(L(m)),$$

where  $D(L(m)) = L(m)((a-c-L(0)\epsilon)/(1+L(0)+m''+L(m)))^2$ .  $D(L(m))$  is increasing in  $L(m)$  until  $1+L(0)+m''$ .  $L(0) = nk - |S^m|$  and  $m'' = 0$  jointly minimize  $D(L(0))$ , regardless of  $L(m)$ . Hence, if  $K > nk - |S^m|$ , then

$$v_k(S^m) = \begin{cases} |S^m|\left(\frac{a-c-(nk-|S^m|)\epsilon}{1+nk}\right)^2 & \text{if } |S^m| \leq 1+nk-|S^m| \\ (nk+1-|S^m|)\left(\frac{a-c-(nk-|S^m|)\epsilon}{2(nk+1-|S^m|)}\right)^2 & \text{if } |S^m| > 1+nk-|S^m|. \end{cases}$$

If  $S^m = N_k$ , then  $v_k(S^m)$  is rewritten with (6) as

$$v_k(S^m) = \max_{1 \leq L(m) \leq nk} L(m)\left(\frac{a-c}{L(m)+1}\right)^2 = \left(\frac{a-c}{2}\right)^2. \quad Q.E.D.$$

## Proof of Theorem 2

Consider first the core with symmetric payoffs to firms, i.e., the set of payoff vectors  $w = (w_0, w_1, \dots, w_{nk}) \in \text{Core}(v_k)$  that assigns to firms the same amount.  $x_i = \sum_{j \in N_k(i)} w_j = kw_j$  is such a core payoff of type  $i \in N$ . If it was nonempty, the following would hold:

$$\begin{aligned} x_0 + s(x_i/k) &\geq v_k(S^0 \cup \{0\}) & s = 1, \dots, nk - 1 \\ nx_i &\geq v_k(N_k), \end{aligned}$$

where  $s = |S^0|$  and  $x_0 + nk(x_i/k) = v_k(N'_k)$ . Then, we have

$$v_k(N'_k) - v_k(S^0 \cup \{0\}) \geq (nk - s)(x_i/k) \geq v_k(N_k) - (s/k)x_i.$$

For any  $s < \infty$ , we have that as  $k$  tends to infinity,

$$\frac{1}{n}(v_k(N'_k) - v_k(S^0 \cup \{0\})) \geq x_i \geq \frac{1}{n}(v_k(N_k)).$$

When  $nk - 1 > 2K$ ,  $v_k(S^0 \cup \{0\}) = K((a - c + \epsilon)/(1 + K))^2 = \epsilon(a - c)$  if  $K \leq s \leq nk - K - 1$ . Hence, for such  $s$ , we have

$$\frac{1}{n}\left(\left(\frac{a - c + \epsilon}{2}\right)^2 - \epsilon(a - c)\right) \geq x_i \geq \frac{1}{n}\left(\left(\frac{a - c}{2}\right)^2\right),$$

but there is no feasible  $x_i$  since  $(\frac{a-c+\epsilon}{2})^2 - (\frac{a-c}{2})^2 < \epsilon(a - c)$ . Hence, the limit core with symmetric payoffs to firms is nonempty.

Suppose next that the limit core was nonempty. Then, there would be a payoff vector in the core that assigns the same amount to identical firms, as is well known. Such a payoff vector does not exist, as shown above. Therefore, the limit core is empty. *Q.E.D.*

## Proof of Theorem 3

Let  $x \in R^{nk}$  be a vector of payoffs to all the firms such that  $x_j = (1/nk)((a - c)/2)^2$  for each firm  $j$ . A pair  $(N_k, x)$  is an objection of an arbitrary firm  $j \in N_k$  against the patent holder to  $\lim_{k \rightarrow \infty} \text{Sh}(v_k)$ , since for each type  $i \in N$  and any integer  $k$ ,  $\sum_{j \in N_k(i)} x_j = (1/n)((a - c)/2)^2 > (1/n)((a - c - \epsilon)/2)^2 = \lim_{k \rightarrow \infty} \text{Sh}_i(v_k)$  and every firm  $j$  receives the same amount in  $\lim_{k \rightarrow \infty} \text{Sh}(v_k)$ .

Since  $\lim_{k \rightarrow \infty} \text{Sh}_0(v_k) = \epsilon(a - c) \geq v_k(S^0 \cup \{0\})$  if  $s \leq 1 + nk - K$  and  $nk - 1 > 2K$ , any possible counterobjection  $(T, y)$  by the patent holder should be made with  $s$  firms, where  $2 + nk - K \leq s$ . Since  $2 + nk - s \leq K$ ,  $v_k(S^0 \cup \{0\}) - sx_j = (a - c + (1 + nk - s)\epsilon)^2 / 4(1 + nk - s) - (s/nk)((a - c)/2)^2$  is increasing in  $s$  for any sufficiently large  $k$ . Hence, let  $s = nk - 1$ . Then,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left\{ \frac{(a - c + (1 + nk - s)\epsilon)^2}{4(1 + nk - s)} - \epsilon(a - c) - \frac{s}{nk} \left(\frac{a - c}{2}\right)^2 \right\} \\ &= \lim_{k \rightarrow \infty} \left\{ \frac{(a - c + 2\epsilon)^2}{8} - \epsilon(a - c) - \frac{n - (1/k)}{n} \left(\frac{a - c}{2}\right)^2 \right\} \\ &= \frac{(a - c + 2\epsilon)^2}{8} - \epsilon(a - c) - \left(\frac{a - c}{2}\right)^2 < 0, \end{aligned}$$

since  $a - c > \epsilon$ . This implies that the patent holder cannot make any counterobjection to the objection  $(N_k, x)$  as  $k$  tends to infinity. *Q.E.D.*

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