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Pricing of Annuities with Surrender Options, Level Premiums and Minimum Guaranteed Interest Rates

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1 introduction

It is important for both policyholders (or customers) and insurance companies to analyze properties of insurance products, and to accumulate their technical knowledge. Since insurance products which transfer and diversify various kinds of risks are necessary for the customers surrounded by uncertainty, information and evaluation standard for insurance products must be provided to the customers. Similarly, since insurance can be regarded as liabilities, analyzing liabilities is essential for the insurance companies. Thus risk-evaluation, risk-management, and asset liability management (ALM) are their urgent tasks.

Iwaki and Yumae (2002) analyzed a corporate pension as such a contingent claim in which there exists a minimum guaranteed interest rate, dividend is if investment performance of the insurance company is good, or the underlying asset is dependent on a trading strategy of the insurance company.

In this paper, we analyze an insurance product of individual annuities. In addition to the above characteristics of the corporate pensions, it has two more characteristics such that the premium is level payment and the policyholders can surrender the contract. Since this contingent claim has a similar characteristic to the American options, we need to analyze the optimal surrender (or exercise) for the policyholders.

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Some studies treated surrender option of insurance products in the literature. Albizzati and Hélyette (1994) evaluated a surrender option of an 8-year-maturity single premium deferred annuity considering the French tax system. They obtained an analytical solution of the surrender option under an assumption that the insurance company could trade bonds that follow the Heath-Jarrow-Morton model. However, since they represented the surrender rate as a piece-width linear function of the ratio between the value when the policyholder surrendered and the value when he or she continued to hold the policy, they did not solved the problem of optimal surrender.

Grosen and Jørgensen (2000) valued an insurance product with lump sum payment, dividends and guaranteed interest rates. After classifying the liability of an insurance company as a reserve for dividends or a policy-reserve, they made a model which decides the dividend amounts based on the reserve for dividends to the policy-reserve and the reserve for dividends. By assuming that the reference portfolio followed the geometric Brownian motion under the risk neutral probability measure, they derived a distribution of terminal share of the policyholders and optimal surrender points numerically by using the Monte Carlo methods and the binomial trees since analytical derivation seems difficult.

Bacinello (2002) evaluated an endowment with a surrender option, lump-sum payment, dividend payments, and minimum guaranteed interest rates, whose contract continues several years. As considering guaranteed interest rates in her model, she made a model where the amount insured was revised by reflecting the maximum between the return of the reference portfolio and the guaranteed interest rate. She represented paths of return of the reference portfolio which fluctuate stochastically as a recombining $(n+1)$-polynomial tree. Under these assumptions, she derived value of the surrender option analytically. Furthermore, she made comparative statics numerically and estimated necessary parameters to design the products.

2 The Model

We consider a continuous-time market model where every asset-price processes are described by continuous-time stochastic processes. The market is assumed to be perfectly competitive and frictionless, and there exists no arbitrage opportunity. The instantaneous interest rate r is assumed to be a nonnegative constant. It is further assumed that there is a reference portfolio for an insurance company. The price process $\{S(t) = \mathrm{e}^{X(t)};\ t \geq 0\}$ of the reference portfolio is assumed to follow such a geometric Brownian motion that

$$X(t) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t); \quad t \geq 0, \tag{1}$$

where $\{W(t);\, t \geq 0\}$ is an Wiener process, and μ and σ are positive constants.

We analyze an annuity issued by the insurance company. Current time is assumed to be time-0. The maturity time T of the annuity is a positive constant $t_n > 0$. Premium payments of the annuity follow a level premium method. That is, a customer pays \$1 as the premium for the annuity at each time t_j, $j = 1, \cdots, n-1$, where $t_{j+1} - t_j = \tau$ is a positive constant, and $t_0 = 0$. Let $r_g \in [0, r)$ denotes the guaranteed interest rate of the annuity which is a constant given at time-0 according to the contract. The sum of amounts $\sum_{j=1}^{n} \mathrm{e}^{jr_g\tau}$ of the principal and the interest accrued until the maturity is guaranteed to be paid back to the customer if he/she does not surrender the annuity until the maturity. In addition to the amounts, at the maturity, an amount $\left(\beta S(T) \sum_{j=1}^{n} \frac{1}{S(t_{j-1})} - \sum_{j=1}^{n} \mathrm{e}^{jr_g\tau}\right)^+$ is paid to the customer as a bonus, where $\beta \in [0, 1]$, which is given at time-0 according to the contract, denotes the participating ratio of customer to the reference portfolio. That is, the bonus is paid if the return of the reference portfolio outperforms the guaranteed interest rate. The customer is assumed to be given right to surrender the annuity at each time $t_1, t_2, \ldots, t_{n-1}$ before paying \$1 of the premium. For given an interest rates r, a guaranteed rate r_g and a participating ratio β, let $\mathcal{C}(t, S(t); r, r_g, \beta)$ denote a time-t value of the annuity before paying \$1 of the premium when price of the reference portfolio is $S(t)$ and the customer selects to continue the annuity. We assume that, for each time t_j, $j = 1, \cdots, n-1$, the customer continues the annuity if

$$\mathcal{C}(t_j, S(t_j); r, r_g, \beta) - 1 \; > \; \sum_{i=1}^{j} \mathrm{e}^{ir_g\tau},$$

otherwise, he/she surrenders the annuity and receives $\sum_{i=1}^{j} \mathrm{e}^{ir_g\tau}$.

3 The analysis of the $n = 2$ case

In this section, we analyze such the $n = 2$ case. In the case, after buying the annuity at time-0, the customer can decide only once whether he/she surrenders the annuity before the maturity or not. At first, we show that, for a given guaranteed interest rate r_g and a given participating ratio β, there exists such a threshold price $S_1^*(r_g, \beta)$ of the reference portfolio that if, at time t_1, price $S(t_1)$ of the reference portfolio is over the threshold price, then the customer continues the annuity, otherwise he/she surrenders the annuity.

Lemma 3.1 *Let $\mathrm{E}^Q[\cdot]$ be the expectation under the risk-neutral probability measure Q. Then,*

$$\begin{aligned}\mathcal{C}(t_1, x; r, r_g, \beta) &= \mathrm{E}^Q\left[e^{-r\tau}\left(\beta(x+1)\mathrm{e}^{X(\tau)} - \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}\right)1_{\{X(\tau)>\ln S_2^*(x;r_g)\}}\right] \\ &\quad + \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau - r\tau}, \qquad (2)\end{aligned}$$

with

$$S_2^*(x; r_g, \beta) := \frac{\sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}}{\beta(x+1)},$$

and there exists a unique $S_1^*(r_g, \beta) \in I\!\!R_+$ satisfying

$$\mathcal{C}(t_1, S_1^*(r_g, \beta); r, r_g, \beta) - 1 = \mathrm{e}^{r_g\tau} \qquad (3)$$

if and only if $\beta \in (0, \beta^* \wedge 1)$ where

$$(1 + \mathrm{e}^{r_g\tau})\left(1 - \mathrm{e}^{-(r-r_g)\tau}\right) = \mathrm{BS}(\beta^*, \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}),$$

and where $\mathrm{BS}(S, K)$ denotes the Black-Scholes formula of a European call option whose time to maturity is τ, strike price is K, and initial value of the underlying is S.

Proof. If there exists no arbitrage, (2) is trivially held. We can easily show that $\mathcal{C}(t_1, x; r, r_g, \beta)$ is strictly increasing with respect to x. Furthermore, since

$$\begin{aligned}&1 + \mathrm{e}^{r_g\tau} - \lim_{x\to+0}\mathcal{C}(t_1, x; r, r_g, \beta) \\ =\ &1 + \mathrm{e}^{r_g\tau} - \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau - r\tau} - \mathrm{BS}(\beta, \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}) \\ =\ &(1 + \mathrm{e}^{r_g\tau})\left(1 - \mathrm{e}^{-(r-r_g)\tau}\right) - \mathrm{BS}(\beta, \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}),\end{aligned}$$

and since $\mathrm{BS}(\beta, \sum_{j=1}^{2} \mathrm{e}^{jr_g\tau})$ is trivially increasing with respect to β, there exists a unique $S_1^*(r_g, \beta) \in I\!R_+$ satisfying (3) if and only if $\beta \in (0, \beta^* \wedge 1)$. □

From Lemma 3.1, we can readily say that the time-0 value of the annuity before paying \$1 of the initial premium is given by the following equation.

$$
\begin{aligned}
&\mathcal{C}(0,1;r,r_g,\beta) \\
= \; & \mathrm{E}^Q\left[\mathrm{e}^{-2r\tau}\left(\beta \mathrm{e}^{X_2}(1+\mathrm{e}^{X_1}) - \sum_{j=1}^{2} \mathrm{e}^{jr_g\tau}\right) 1_{\{X_2 > \ln(S_2^*(X_1;r_g,\beta)),\, X_1 > \ln(S_1^*(r_g,\beta))\}}\right] \\
&+ \left(\sum_{j=1}^{2} \mathrm{e}^{jr_g\tau - 2r\tau} - \mathrm{e}^{-r\tau}\right) \mathrm{E}^Q[1_{\{X_1 > \ln(S_1^*(r_g,\beta))\}}] \\
&+ \mathrm{e}^{-(r-r_g)\tau}\mathrm{E}^Q\left[1_{\{X_1 \le \ln(S_1^*(r_g,\beta))\}}\right], \qquad (4)
\end{aligned}
$$

where $X_1 \overset{\mathrm{d}}{=} X(t_1)$ and $X_2 \overset{\mathrm{d}}{=} X(t_2) - X(t_1)$. From (1), noting X_1 and X_2 are mutually independent and follow the same normal distribution with mean $\left(r - \frac{1}{2}\sigma^2\right)\tau$ and variance $\sigma^2\tau$ under the risk-neutral measure Q, we can explicitly represent (4) as

$$
\begin{aligned}
&\mathcal{C}(0,1;r,r_g,\beta) \\
= \; & \beta\left\{\int_{-d_1(1/S_1^*(r_g,\beta))}^{\infty} \Phi\left(d_1\left(\frac{\beta\left(1+\mathrm{e}^{\left(r+\frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}z}\right)}{\sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}}\right)\right)\mathrm{d}\Phi(z)\right. \\
&\left. + \mathrm{e}^{-r\tau}\int_{-d_2(1/S_1^*(r_g,\beta))}^{\infty} \Phi\left(d_1\left(\frac{\beta\left(1+\mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}z}\right)}{\sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}}\right)\right)\mathrm{d}\Phi(z)\right\} \\
&- \sum_{j=1}^{2}\mathrm{e}^{jr_g\tau - 2r\tau}\int_{-d_2(1/S_1^*(r_g,\beta))}^{\infty} \Phi\left(d_2\left(\frac{\beta\left(1+\mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}z}\right)}{\sum_{j=1}^{2}\mathrm{e}^{jr_g\tau}}\right)\right)\mathrm{d}\Phi(z) \\
&+ \left(\sum_{j=1}^{2}\mathrm{e}^{jr_g\tau - 2r\tau} - \mathrm{e}^{-r\tau}\right)\Phi(d_2(1/S_1^*(r_g,\beta))) \\
&+ \mathrm{e}^{-(r-r_g)\tau}\Phi(-d_2(1/S_1^*(r_g,\beta))), \qquad (5)
\end{aligned}
$$

where

$$
d_1(x) := \frac{\ln x + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2(x) := d_1(x) - \sigma\sqrt{\tau},
$$

and where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

Next proposition is our main result which says that there exists a unique combination of a guaranteed interest rate r_g and a participating ratio β balancing between payment and receipt amounts for the customers.

Proposition 1 *There exists such a unique combination of a* $r_g \in [0, r)$ *and a* $\beta \in (0, \beta^* \wedge 1)$ *that*

$$\mathcal{C}(0,1;r,r_g,\beta) = 1. \tag{6}$$

Proof. . A tedious algebra leads that

$$\frac{\partial}{\partial r_g}\mathcal{C}(0,1;r,r_g,\beta) > 0, \quad \frac{\partial}{\partial \beta}\mathcal{C}(0,1;r,r_g,\beta) > 0$$

and that

$$\lim_{r_g \to +0, \beta \to +0} \mathcal{C}(0,1;r,r_g,\beta) < 1.$$

Furthermore, since

$$\lim_{r_g \uparrow r} \mathcal{C}(0,1;r,r_g,\beta) > 1 \quad \forall \beta \in (0,1),$$

we obtain the desired result. □

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