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The Value of Collusion in Multimarket Contact

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Abstract

This paper investigates the effect of multimarket contact on the value of collusion under imperfect monitoring. We show that firms can improve the value of collusion via multimarket contact by comparing this value with the sum of the values achieved in the case of single market contact. Specifically, firms can avoid efficiency loss in markets with less tempting deviation through multimarket contact. We clearly express the relation between the value of collusion in multimarket contact and the degree of temptation to deviate through the formula by Abreu, Milgrom and Pearce (1991).

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1 Introduction

Modern firms diversify their businesses; even within one business unit, firms may have multiple product lines and geographically separate sales areas. Consequently, firms compete with the same rivals in various markets. For example, airline companies compete with their rivals in terms of various routes. For the 1000 largest routes in the US airline market, American and Delta operated 527 of the same routes, American and Northwest operated 357 of the same routes, and Delta and Northwest operated 323 of the same routes in 1988 (see Evans and Kessides (1994)).

Many researchers advocate that collusion may be easier to sustain when firms interact with each other in many markets. For example, Edwards (1955) states that multimarket contact makes collusive agreement easier to sustain because a local deviation from collusion leads to retaliation in all markets.[1] Bernheim and Whinston (1990) were the first to give a theoretical foundation for the relationship between collusion and multimarket contact. They state that multimarket contact is not always effective in the stability of collusion. Specifically, they show that multimarket contact does not aid in sustaining collusive outcomes when identical firms meet in identical markets, which is *the irrelevance result*. In addition, Bernheim and Whinston (1990) provide the circumstances under which multimarket contact assists in sustaining collusive outcomes, such as market heterogeneity in the number of firms, the growth rates, the response lags, demand fluctuations, and so on. This heterogeneity makes collusion more stable by transferring enforcement powers across markets.

The study of Bernheim and Whinston (1990) depends on the assumption of perfect monitoring.[2] Therefore, their main focus is how stable collusion can be. However, as suggested by the seminal work of Green and Porter (1984), we cannot always expect firms to observe their rivals' actions, because in some industries prices are negotiated between firms and customers on a case-by-case basis. For this reason, it is difficult to monitor a rival's secret price-cutting.

If we assume imperfect monitoring, this assumption raises a new issue: the degree of value that is generated by collusion. This is because imperfection in monitoring technology may cause an efficiency loss (see Radner, Myerson and Maskin (1986), and Fudenberg, Levine and Maskin (1994)). We will investigate the effect of multimarket contact on the sustainability of the highest possible value of collusion under imperfect monitoring. By applying the formula introduced by Abreu, Milgrom and Pearce (1991) (hereafter we call this formula the AMP formula), we show that if firms are sufficiently patient, firms can improve the value of collusion *via* multimarket contact by comparing this value with the sum of the values achieved in the single market contact situation.[3] Specifically, through multimarket contact, firms can fully avoid the efficiency loss except in the market with the most *tempting* deviation, which leads to more gain and is less detectable.

The key aspects of this assertion are (i) that multimarket contact generates additional information, and (ii) that multimarket contact makes punishment more severe. Firms would be able to use multiple signals from multiple markets. This means that more information on a rival's behavior is acquired by multimarket contact than by sin-

[1] Evans and Kessides (1994) empirically test Edwards's statement using US airline industry data.

[2] Spagnolo (1999a) also analyzes the effect of multimarket contact on the stability of collusion under perfect monitoring. He shows that if payoffs of firms are strictly concave in their profits, then multimarket contact always relaxes the incentive compatibility conditions.

[3] Sekiguchi (2001) surveys the AMP formula and refers to the implications of the formula in private monitoring.

gle market competition. The many signals acquired enable firms to conduct statistical inferences more accurately on the rival's deviations from collusion.

Furthermore, based on a statement by Edwards (1955), when firms have contact with each other in multiple markets, a local deviation from collusion leads to retaliation in all markets, that is *global retaliation*. In combination with accurate statistical inferences, firms hesitate to deviate in the markets with less tempting deviation for fear of global retaliation. Hence, the combination of accurate inferences and the threat of global retaliation alleviates the efficiency loss in those markets while sustaining firms' incentives to collude in all markets.[4]

The result has implications for competition policy in implicit collusion. We can confirm that the multimarket contact situation may be a hotbed of collusion under both perfect monitoring and imperfect monitoring. Therefore, governments must be aware of multimarket contact so as to detect collusion. In addition, we are able to derive an explicit formula to measure the value of collusion, which can be verified by empirical research.

The results in this paper are applicable to many economic problems. First, we can explain the role of multilateral relationships in repeated partnerships. An example is that modern firms offer various multilateral relationships to their employees. A typical case is that employees are sometimes engaged in a multitask partnership. Furthermore, Japanese firms often rent company-owned houses to their employees. This induces employees to interact with each other outside the workplace. Another example is a mutual aid association in rural communities. Spagnolo (1999b) shows multilateral relationships foster cooperation under perfect monitoring. In this respect, we extend the analysis of Spagnolo (1999b) to the situation of imperfect monitoring and show that multilateral relationships among members in an organization improve the value of cooperation under imperfect monitoring.

Second, the results obtained in this paper have new implications in the literature of *umbrella branding* or brand stretching. Umbrella branding is the labeling of more than one product with a single brand name. Suppose that customers can observe the brand name but not the identity of each brand owner. Subsequently, the situation is the same as that of a multimarket contact setting, when the firm adopts umbrella branding and the set of customers sufficiently overlaps among products. Cabral (2007) considers the issue in the framework of repeated-purchase of experience goods with the seller's moral hazard. In his model, a firm chooses the levels of quality for its products, and subsequently consumers cannot observe these levels of quality but infer them from the performance of the products. That is, the performance of the products plays the role of imperfect public signals. He shows that in the case of two products with identical signals of quality, if the firm is sufficiently patient, umbrella branding is superior to no umbrella branding.[5] We extend this result to an arbitrary number of products with general distribution of signals. We can clearly express the degree of superiority of umbrella branding and the optimal brand size.

Finally, the results have an implication for policy coordination among nations in areas such as trade policy, macroeconomic policy, and global environmental issues. Spagnolo (2001) investigates the optimal design of self-enforcing international policy coordination in a multi-issue context. He attempted to determine when it is beneficial

[4]In contrast to the irrelevance result of Bernheim and Whinston (1990), our result holds whether the markets are heterogeneous or not.

[5]Cabral (2007) characterizes the optimal equilibrium for any discount factor, while we focus on a sufficiently large discount factor.

to governments to link multiple policy issues. However, his analysis is based on the assumption of perfect monitoring. For some policy issues imperfect monitoring is a more reasonable assumption than perfect monitoring. A prominent example, introduced by Riezman (1991), is a policy agreement to eliminate or reduce nontariff trade barriers. We shed light on linking multiple issues with imperfect observability.

This paper is not the first to examine the effect of multimarket contact on collusion under imperfect monitoring. Matsushima (2001) analyzes multimarket contact under imperfect monitoring in the case of identical information structures across the markets. He shows that under a fixed and possibly low discount factor such as that in perfect monitoring, we can obtain an efficient outcome if the number of contacts is sufficiently large. This is because when the number of contacts is sufficiently large, firms can almost certainly detect a rival's deviations by the "law of large numbers." This logic depends on the following two assumptions: (i) information structures across the markets are identical, and (ii) the number of contacts is infinitely large. However, in the real world, there are cases in which it is difficult for conglomerate firms to diversify in a manner that satisfies these assumptions.

In contrast to the aforementioned analysis, we study the influence of multimarket contact on collusion under imperfect monitoring in general information structures given an arbitrary number of contacts. By focusing on the case in which firms are sufficiently patient, we succeed in clarifying the relationship between the degree of temptation to deviate and the value of collusion using the AMP formula. Using this formula, we can confirm that additional information acquired by multimarket contact improves the ability to detect an opponent's deviations without using the law of large numbers. Even in the case where firms come into contact with each other in only two markets, multimarket contact may improve the value of collusion under imperfect monitoring.

In the sense that additional information improves the value of cooperation, this paper is related to the use of private strategies explored by Kandori and Obara (2006). In their paper they utilize private histories as additional information, whereas in our paper we utilize public histories in additional markets.

Recently, the closely related research by Cai and Obara (2006) independently develops a model of firm reputation and horizontal integration, which is related to umbrella branding.[6] [7] They characterize the optimal degree of integration. Specifically, they show that efficiency loss might be alleviated by diversifying relations with customers under certain conditions, and clarify that a bound on the firm size may naturally arise. Although these results are very similar to ours, their paper and ours differ in two respects. First, they focus on a one-sided moral hazard problem to study firm reputation in the interaction between a long-lived firm and short-lived customers, while we analyze the multiple-sided moral hazard problem to study the collusion among long-lived firms. As a result, we discuss the optimal equilibrium including asymmetric strategies, which is absent from their paper. Second, they consider identical relationships between a firm

[6]We thank Luis Cabral and Tadashi Sekiguchi for informing us that this paper is closely related to ours.

[7]Several studies explore the relation between imperfect information and the effectiveness of multimarket contact, although the settings are somewhat different from the present one. Kandori and Obara (2004) study the relation under repeated multitask partnerships with costly monitoring. They show that, regardless of how large the monitoring cost per task is, the efficient outcome can be approximated as players become patient when there are many tasks. Kobayashi and Ohta (2007) investigate multimarket contact in continuous time games. By applying the result of Sannikov (2007), they show that for any discount rate, the set of equilibrium payoffs per market strictly expands toward the direction of improving efficiency through multimarket contact.

and customers across the markets, while we consider more general relationships among firms including heterogeneous ones across the markets with respect to (i) cooperative payoffs, (ii) deviation gain from collusion, and (iii) information structure. As a result, we can understand the effect of the heterogeneous market structure on collusion.

This paper is organized as follows. In section 2, we describe the model of duopoly. In section 3, we introduce the AMP formula and investigate the value of collusion in multimarket contact by focusing on symmetric equilibrium. In section 4, we extend the analysis by incorporating mixed and asymmetric strategies. Then we show that under a sufficient condition, the value of collusion derived in the previous section is an upper bound of equilibrium payoffs among any perfect public equilibrium for any discount factor. In section 5, we make concluding remarks and discuss possible avenues of future research. We discuss the extension to the n-firm case in the Appendix.

2 The Model

Consider two identical firms (firm 1 and 2) competing with each other in m markets over infinite periods $t = 0, 1, 2, \cdots$.[8] Each firm has binary choices, "to cooperate with the opponent (C)" or "to defect from cooperation (D)" for each market. We denote the set of choices for firm i in market k by $A_i^k = \{C, D\}$. Let $A_i = \prod_{k=1}^m A_i^k$, $A^k = A_1^k \times A_2^k$, and $A = A_1 \times A_2 = \prod_{k=1}^m A^k$.

Firms cannot observe the rival's actions but receive imperfect public signals of the actions. Let $\Omega^k \equiv \{G, B\}$ be the set of the signals for market k, and let $\Omega = \Pi_{k=1}^m \Omega^k$ be the product. Here, G represents a "good outcome" whereas B stands for a "bad outcome." This is a situation in which firms cannot achieve full efficiency even if a firm has a long-term relationship with the rival in each market.[9] In other words, we consider the model in which imperfection of the monitoring technology becomes a serious obstacle for firms to collude in the sense that efficiency loss is inevitable. With this model, we focus on how multimarket contact alleviates the obstacle.

For each k, given an action profile $\boldsymbol{a^k} = (a_1^k, a_2^k) \in A^k$, a signal $\omega^k \in \Omega^k$ is realized with probability $P^k(\omega^k|\boldsymbol{a^k})$. In the multimarket contact situation, given an action profile $\boldsymbol{a} = (a_1^1, a_1^2, \cdots, a_1^m; a_2^1, a_2^2, \cdots, a_2^m) \in A$, a signal vector $\boldsymbol{\omega} = (\omega^1, \omega^2, \cdots, \omega^m) \in \Omega$ is realized with probability $P(\boldsymbol{\omega}|\boldsymbol{a})$. We focus on the case in which the public signals of each market are independently distributed and are not influenced by the actions taken in the other markets. Formally, the assumption is as follows.

Assumption 1. $P(\boldsymbol{\omega}|\boldsymbol{a}) = \prod_{k=1}^m P^k(\omega^k|\boldsymbol{a^k})$.

For notational convenience, let $p_2^k \equiv P^k(G|C, C)$, $p_1^k \equiv P^k(G|D, C) = P^k(G|C, D)$, and $p_0^k = P^k(G|D, D)$. Then, we make the following standard assumption to focus on interesting and natural situations.

Assumption 2. $1 > p_2^k > p_1^k > p_0^k \geq 0$ *for all* $k = 1, \cdots, m$.

Assumption 2 guarantees that a unilateral defection decreases the probability of the good outcome.

[8] We can extend the analysis to the case of n firms. See the Appendix.

[9] Our model does not satisfy the pairwise full rank condition, which is a sufficient condition for the folk theorem by Fudenberg, Levine and Maskin (1994). Because of the failure of pairwise full rank, we cannot attribute one bad signal to a particular firm's deviation. Thus firms must punish each other upon realization of a bad outcome, which causes unavoidable efficiency loss. See also Radner, Myerson and Maskin (1986).

Each firm obtains profit $r_i^k = r_i^k(a_i^k, \omega^k)$ from market k when it takes $a_i^k \in A_i^k$ and ω^k is realized. We denote the expected payoffs for firm i from market k by $u_i^k(\boldsymbol{a^k}) = \sum_{\omega^k} P^k(\omega^k|\boldsymbol{a^k}) r_i^k(a_i^k, \omega^k)$. In total, each firm obtains $u_i(\boldsymbol{a}) = \sum_{k=1}^m u_i^k(\boldsymbol{a^k})$ in each stage game.

We assume that the stage expected payoff structure in any market k has the property of a symmetric Prisoners' Dilemma.[10] This represents a symmetric Cournot competition in a simplified manner. The following assumption guarantees that (C, C) is the efficient action profile for each k, while $a_i^k = D$ is the dominant action for each i, k.

Assumption 3. *$\pi^k > 0,\ g^k > 0,\ b^k > 0$ and $g^k - b^k < \pi^k$.*

	C	D
C	π^k, π^k	$-b^k, \pi^k + g^k$
D	$\pi^k + g^k, -b^k$	$0, 0$

Table 1: Prisoners' Dilemma

In each period, the stage game is played and then the corresponding public signal is revealed. We assume that a public randomization device is available at the end of each period. The randomization device selects a number $\gamma \in [0, 1]$ according to the uniform distribution on $[0, 1]$. The public history at the beginning of period t is represented by a sequence $h(t) = \{\boldsymbol{\omega}(t'), \gamma(t')\}_{t'=0}^{t-1}$. The private history at the beginning of period t is $h_i(t) = \{\boldsymbol{a_i}(t')\}_{t'=0}^{t-1}$. We denote the set of all public histories and private histories for firm i by H^t and H_i^t, respectively. Let $\mathcal{H}_i = \bigcup_{t=0}^{\infty}(H^t \times H_i^t)$. A strategy of firm i, σ_i, is a map from $\mathcal{H}_i$ to the set of (randomized) action. We call Σ_i the set of all firm i's strategies, and $\Sigma = \Sigma_1 \times \Sigma_2$ the set of all strategy profiles. Given a common discount factor $\delta \in (0, 1)$ and a sequence of action profiles, $\{\boldsymbol{a}(t)\}_{t=0}^{\infty}$, generated by a strategy profile $\boldsymbol{\sigma} \in \Sigma$, the firm i's average discounted expected profit is $(1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(\boldsymbol{a}(t))$.

We focus on *perfect public equilibria.* A strategy of player i is *public* if at each time, t, it depends on the public history $h(t)$ and not on the private history $h_i(t)$. A perfect public equilibrium is a profile of public strategies such that at every date t and for any public history $h(t)$ the strategies constitute a Nash equilibrium from that date onwards. Note that in a perfect public equilibrium the players' beliefs about the opponents' past play are irrelevant. Therefore, perfect public equilibria are a special class of sequential equilibria.

3 Analysis

3.1 Measuring the value of collusion

We study how multimarket contact enhances the value of cooperation. Specifically, we will compute the equilibrium maximizing the sum of payoffs in multimarket contact by the method of AMP.

[10] We can analyze an asymmetric Prisoners' Dilemma by allowing firms to propose shares of total profits. An example of a share is market share in a price setting oligopoly as in Athey and Bagwell (2001). Kobayashi, Ohta and Sekiguchi (2007) characterize the optimal sharing rules that sustain the most efficient outcome for any discount factor in repeated partnerships.

The method of AMP corresponds to seeking the best equilibrium value in strongly symmetric perfect public equilibrium of a symmetric game. A public strategy profile is *(strongly) symmetric* if at each time t and for any public history $h(t)$ the strategies prescribe the same action for both firms. Following AMP, we also restrict our attention to pure strategy for the moment. In section 4, we will discuss mixed strategies and give a sufficient condition that the value derived by the method of AMP is the best among any perfect public equilibria.

To derive the AMP formula in our model, we limit our attention to a generalized trigger strategy, and compute the highest value among them. A generalized trigger strategy has two phases: C phase and D phase. In C phase each firm plays a "co-operative" action, while each firm plays the dominant action in D phase. The phase in period 1 is C phase. If period $t-1$ is in C phase and if the signal in period $t-1$ is $\boldsymbol{\omega}$, then the phase switches to D with probability $\rho_{\boldsymbol{\omega}} \in [0,1]$. With the remaining probability, the phase remains in phase C. If period $t-1$ is in D phase, then period t is in D phase with certainty. That is, D phase is the absorbing phase. Thus, the problem is described as:

$$\max_{\boldsymbol{a},\boldsymbol{\rho}} \ v$$

s.t.

$$v = (1-\delta)u(\boldsymbol{a}) + \delta \sum_{\boldsymbol{\omega}\in\Omega} \prod_{k=1}^{m} P^k(\omega^k|\boldsymbol{a^k})(1-\rho_{\boldsymbol{\omega}})v, \tag{1}$$

$$v \geq (1-\delta)u_i(\boldsymbol{\hat{a}_i}, \boldsymbol{a_j}) + \delta \sum_{\boldsymbol{\omega}\in\Omega} \prod_{k=1}^{m} P^k(\omega^k|\hat{a}_i^k, a_j^k)(1-\rho_{\boldsymbol{\omega}})v \tag{2}$$

$$\text{for all } \boldsymbol{\hat{a}_i} \in A_i, \ i = 1, 2.$$

Equation (1) represents the value function and Inequality (2) represents the incentive compatibility conditions.

Before analyzing the multimarket contact case, we introduce the value of collusion in a single market contact situation using the AMP formula. This is a benchmark and is a special case of the multimarket contact case. Given an arbitrary market k, the value functions and the incentive conditions are:

$$v^k = (1-\delta)u^k(\boldsymbol{a}^k) + \delta\{P^k(G|\boldsymbol{a}^k)(1-\rho_G) + (1-P^k(G|\boldsymbol{a}^k))(1-\rho_B)\}v^k,$$
$$v^k \geq (1-\delta)u_i^k(\hat{a}_i^k, a_j^k) + \delta\{P^k(G|\hat{a}_i, a_j^k)(1-\rho_G) + (1-P^k(G|\hat{a}_i, a_j^k))(1-\rho_B)\}v^k,$$

for all $\hat{a}_i^k \in A_i^k$. To derive the highest value, the incentive condition must be binding at some deviation action $\hat{a}_i^k$. By arranging the value function and the incentive condition, we can obtain the following formula:

$$v^k = u^k(\boldsymbol{a}^k) - \frac{u_i^k(\hat{a}_i^k, a_j^k) - u^k(\boldsymbol{a}^k)}{\frac{(1-P^k(G|\hat{a}_i, a_j^k))\rho_B + P^k(G|\hat{a}_i, a_j^k)\rho_G}{(1-P^k(G|\boldsymbol{a}^k))\rho_B + P^k(G|\boldsymbol{a}^k)\rho_G} - 1}.$$

When we set $\boldsymbol{a}^k = (C, C)$, the value is maximized with $\rho_B > 0$ and $\rho_G = 0$, because $p_2^k > p_1^k$. Therefore, the maximized value is:

$$v^{*k} = \pi^k - \frac{g^k}{\frac{1-p_1^k}{1-p_2^k} - 1}.$$

If the value is positive, then it is the highest value of the equilibria. On the other hand, if it is negative, then the repetition of the Nash equilibrium of the stage game is optimal and the value is zero. Let us denote $l^k \equiv \frac{1-p_1^k}{1-p_2^k}$ for notational convenience. Then, we can summarize the argument as follows.

Proposition 0 (Abreu, Milgrom and Pearce (1991)). *Suppose that Assumptions 2 and 3 hold. If $g^k < \pi^k(l^k - 1)$, then there exists $\underline{\delta} \in (0,1)$ such that for any $\delta \geq \underline{\delta}$ the optimal symmetric equilibrium value in market k is:*

$$v^{*k} = \pi^k - \frac{g^k}{l^k - 1}. \tag{3}$$

If $g^k \geq \pi^k(l^k - 1)$, it is zero for any δ.

Equation (3) is the AMP formula and we call the level of the value the AMP value. Note that this formula takes the form of a cooperative payoff minus efficiency loss attributed to imperfect monitoring. Because the component of the efficiency loss in the formula depends on the deviation gain and informativeness of signals, we can analyze the relationship among value, deviation gain, and informativeness.

3.2 Collusion in all markets

Let us return to the analysis of multimarket contact. We will compute the highest possible value of collusion in all markets. In the case of multimarket contact, we must take the following new issues into account. First, there are many signal profiles. This leads to difficulty in determining when firms trigger. Second, we must consider a great variety of deviations. Consequently, it is difficult to explore which incentive condition is binding. If we, without loss of generality, permutate the index of the markets as $\frac{g^1}{l^1-1} \geq \frac{g^2}{l^2-1} \geq \cdots \geq \frac{g^m}{l^m-1}$, we obtain the following result.

Proposition 1. *Suppose that Assumptions 1, 2, and 3 hold. If $g^1 < \pi^1(l^1 - 1)$, then there exists $\underline{\delta} \in (0,1)$ such that for any $\delta \geq \underline{\delta}$ the optimal symmetric equilibrium value under multimarket contact is:*

$$v^* = \sum_{k=1}^{m} \pi^k - \frac{g^1}{l^1 - 1}.$$

Proof. We will prove the proposition by the following three steps: (i) for any deviation in C phase we will seek a sequence $\{\rho_{\boldsymbol{\omega}}\}_{\boldsymbol{\omega}\in\Omega}$ that maximizes the equilibrium value; (ii) given the sequence, we will find the binding incentive condition; and (iii) we will derive the best value from the value equation and the binding incentive condition.

Let $\boldsymbol{a}^{*k} = (C, C)$ be the most efficient action profile in market k. Let $\boldsymbol{a}^* = (\boldsymbol{a}^{*1}, \boldsymbol{a}^{*2}, \cdots, \boldsymbol{a}^{*m})$. Suppose firms choose $\boldsymbol{a}^*$ in C phase. For an arbitrary action $\hat{a}_i \in A_i$, let $l_{(\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)|\boldsymbol{a}^*}(\boldsymbol{\rho}) \equiv \frac{\sum_{\boldsymbol{\omega}\in\Omega}\prod_{k=1}^m P^k(\omega^k|(\hat{a}_i^k, a_j^{*k}))\rho_{\boldsymbol{\omega}}}{\sum_{\boldsymbol{\omega}\in\Omega}\prod_{k=1}^m P^k(\omega^k|\boldsymbol{a}^{*k})\rho_{\boldsymbol{\omega}}}$ be the likelihood ratio of signals with respect to the deviation $\hat{a}_i$ when firms are playing $\boldsymbol{a}^*$.

Consider a unilateral deviation from $\boldsymbol{a}^*$ by firm i. Fix the deviation action $\hat{\boldsymbol{a}}_i$. If we make the incentive condition for the deviation as an equality, we can obtain the following equation from the value equation and the incentive condition:

$$v = \sum_{k=1}^{m} \pi^k - \frac{u_i(\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*) - u_i(\boldsymbol{a}^*)}{l_{(\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)|\boldsymbol{a}^*}(\boldsymbol{\rho}) - 1}.$$

Note that $\max_{\boldsymbol{\rho}} l_{(\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)|\boldsymbol{a}^*}(\boldsymbol{\rho}) = \frac{P(\boldsymbol{\omega}=(B,\cdots,B)|\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)}{P(\boldsymbol{\omega}=(B,\cdots,B)|\boldsymbol{a}^*)}$ for any $\hat{\boldsymbol{a}}_i$. That is, the strategy triggers only when bad signals are observed in all markets and attains the best value. This stems from the following arguments. For any $\boldsymbol{\omega'} \neq (B, \cdots, B)$:

$$\frac{P(\boldsymbol{\omega} = (B, \cdots, B)|\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)}{P(\boldsymbol{\omega} = (B, \cdots, B)|\boldsymbol{a}^*)} \geq \max_{\boldsymbol{\omega'} \neq (B,\cdots,B)} \frac{P(\boldsymbol{\omega'}|\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)}{P(\boldsymbol{\omega'}|\boldsymbol{a}^*)},$$

because the right-hand side of the inequality contains the likelihood ratio $\frac{p_1^{k'}}{p_2^{k'}}$ or $\frac{p_2^{k'}}{p_2^{k'}}$, whereas the left-hand side consists of the likelihood ratio $\frac{1-p_1^k}{1-p_2^k}$ or $\frac{1-p_2^k}{1-p_2^k}$ for any k. Because, for any positive real number x, y, $\frac{y}{x} > \frac{y'}{x'} \Leftrightarrow \frac{y}{x} > \frac{y+y'}{x+x'}$ by simple algebra, we can say that:

$$\frac{P(\boldsymbol{\omega} = (B, \cdots, B)|\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)}{P(\boldsymbol{\omega} = (B, \cdots, B)|\boldsymbol{a}^*)} \geq \max_{\boldsymbol{\rho'} \notin \mathcal{A}} \frac{\sum_{\boldsymbol{\omega}} P(\boldsymbol{\omega}|\hat{\boldsymbol{a}}_i, \boldsymbol{a}_j^*)\rho'_{\boldsymbol{\omega}}}{\sum_{\boldsymbol{\omega}} P(\boldsymbol{\omega}|\boldsymbol{a}^*)\rho'_{\boldsymbol{\omega}}},$$

where $\mathcal{A} = \{\boldsymbol{\rho} \in [0,1]^m | \rho_{(B\cdots B)} > 0, \rho_{\boldsymbol{\omega} \neq (B\cdots B)} = 0\}$ by Assumption 2.

If we verify the most profitable deviation under the strategy and show that the strategy is incentive compatible, then the equilibrium value of collusion is obtained given the action profile $\boldsymbol{a}^*$ in C phase. Let us denote $M = \{1, 2, \cdots, m\}$, and consider the case in which a player deviates in markets $M' \subseteq M$. Because, under the optimal trigger strategy, firms adopt the strategy that triggers only when bad signals are observed in all markets, the deviation is unprofitable if:

$$\delta[\prod_{k \in M'} (1 - p_1^k) \prod_{k \in M \setminus M'} (1 - p_2^k) - \prod_{k \in M} (1 - p_2^k)]\rho_{(B\cdots B)} v \geq (1 - \delta) \sum_{k \in M'} g^k$$

Therefore, the most profitable deviation is taking D in markets M' to maximize:

$$\frac{\sum_{k \in M'} g^k}{\prod_{k \in M'} l^k - 1}.$$

We show that the most profitable deviation is taking D in only market 1. First, the following inequality holds:

$$\frac{g^1}{l^1 - 1} > \frac{g^1 + g^2}{l^1 l^2 - 1}. \tag{4}$$

This is because (4) is equivalent to $l^1 \times \frac{g^1}{l^1-1} > \frac{g^2}{l^2-1}$.
Next, we assume that:

$$\frac{g^1 + g^2 + \cdots + g^{k-1}}{l^1 l^2 \cdots l^{k-1} - 1} > \frac{g^1 + g^2 + \cdots + g^k}{l^1 l^2 \cdots l^k - 1}. \tag{5}$$

Then:

$$
\begin{aligned}
(5) \quad \Leftrightarrow \quad & l^1 l^2 \cdots l^{k-1} \frac{(g^1 + g^2 + \cdots + g^{k-1})}{l^1 l^2 \cdots l^{k-1} - 1} > \frac{g^k}{l^k - 1} \\
\Leftrightarrow \quad & (g^1 + g^2 + \cdots + g^{k-1})(l^1 l^2 \cdots l^k - l^1 l^2 \cdots l^{k-1}) > (l^1 l^2 \cdots l^{k-1} - 1) g^k \\
\Leftrightarrow \quad & (g^1 + g^2 + \cdots + g^{k})(l^1 l^2 \cdots l^k - l^1 l^2 \cdots l^{k-1}) > (l^1 l^2 \cdots l^{k} - 1) g^k \\
\Leftrightarrow \quad & l^1 l^2 \cdots l^{k-1} \frac{(g^1 + g^2 + \cdots + g^{k})}{l^1 l^2 \cdots l^{k} - 1} > \frac{g^k}{l^k - 1}. \qquad (6)
\end{aligned}
$$

(6) implies that:

$$
\begin{aligned}
& l^1 l^2 \cdots l^{k} \frac{(g^1 + g^2 + \cdots + g^{k})}{l^1 l^2 \cdots l^{k} - 1} > \frac{g^{k+1}}{l^{k+1} - 1} \\
\Leftrightarrow \quad & \frac{g^1 + g^2 + \cdots + g^{k}}{l^1 l^2 \cdots l^{k} - 1} > \frac{g^1 + g^2 + \cdots + g^{k+1}}{l^1 l^2 \cdots l^{k+1} - 1}.
\end{aligned}
$$

Therefore, combining the above inductive argument with the fact $\frac{g^1}{l^1-1} \geq \frac{g^2}{l^2-1} \geq \cdots \geq \frac{g^m}{l^m-1}$, we have:

$$
\max_{M'} \frac{\sum_{k \in M'} g^k}{\prod_{k \in M'} l^k - 1} = \frac{g^1}{l^1 - 1}
$$

Therefore, the binding incentive condition is the one deterring the deviation in market 1.

Take $\underline{\delta}$ such that:

$$
\underline{\delta}[(1 - p_1^1) - (1 - p_2^1)] \Pi_{k'=2}^{m} (1 - p_2^{k'}) v = (1 - \underline{\delta}) g^1.
$$

For any $\delta > \underline{\delta}$, take $\rho_{(BB \cdots B)} \in (0, 1)$ such that:

$$
\delta[(1 - p_1^1) - (1 - p_2^1)] \Pi_{k'=2}^{m} (1 - p_2^{k'}) \rho_{(B \cdots B)} v = (1 - \delta) g^1.
$$

By the above argument, it is obvious that this strategy constitutes an equilibrium. Therefore, the equilibrium value is:

$$
v^* = \sum_{k=1}^{m} \pi^k - \frac{g^1}{l^1 - 1}.
$$

Finally, if $g^1 < \pi^1 (l^1 - 1)$, then the value is higher than any equilibrium value given $\boldsymbol{a} \neq \boldsymbol{a}^*$ in C phase. ∎

Firms statistically infer deviations under imperfect monitoring. Specifically, firms test whether an opponent deviates or not by using the public signals as samples of the opponent's behavior. Among many types of statistical tests, the most powerful test in this case is the hypothesis that "the opponent deviated" against the hypothesis that "the opponent did not deviate" when observing the bad signals in every market. This is because testing the opponent's behavior in a market by using the good signals does not increase the likelihood ratio of the hypothesis "the opponent deviated" against the hypothesis "the opponent did not deviate." Given the test, when firms are sufficiently patient, firms do not intend to deviate in multiple markets, in which case deviations are easier to detect. If a firm deviates in multiple markets, the likelihood ratio for the deviation increases in an exponential order. Because the gains from multilateral deviations increase in a linear way, the firm hesitates to deviate in multiple markets. Therefore, firms intend to deviate only in the market with the most *tempting* deviation in which the deviation leads to more gain and is less detectable so that it is sufficient to deter the deviation in the market with the most tempting deviation. Therefore, the loss attributed to imperfect monitoring can only be the result of the market with the most tempting deviation. In other words, the efficiency loss can be *fully* alleviated in the other markets so that the value of collusion (per market) improves in a multimarket contact setting.

Note that, in the generalized trigger strategy, firms should take D in all markets in D phase. This means that if a local deviation from collusion leads to bad signals in all markets, the deviation leads to retaliation in all markets. This strategy describes the global retaliation proposed by Edwards (1955). Combining the threat of global retaliation with the statistical inference described above, firms are likely to hesitate to deviate.

From Proposition 1, we can derive what types of multimarket contact are preferable for firms with respect to the degree of temptation to deviate. Following Proposition 1, the less temptation to deviate in market 1, the higher is the value of collusion. In contrast, the more temptation to deviate in markets 2 to m, the greater is the reduction of efficiency loss thanks to multimarket contact. The latter case indicates what determines the degree of improvement of social benefit when committing to multilateral relationships.

Efficiency loss also determines the degree of diversification for firms. Proposition 1 states that firms can attain the best value by colluding in every market, if the loss is sufficiently small relative to cooperative payoffs in market 1. Although colluding in every market can still be enforceable even when $g^1 > \pi^1(l^1-1)$ but $g^1 < \sum_{k=1}^{m} \pi^k(l^1-1)$, such collusion might not achieve the best value. That is, there are cases in which firms may attain the highest value by giving up collusion in markets with more tempting deviation.

Suppose firms collude in every market. In that case, consider a situation in which they give up collusion in markets 1 to k where deviations are more tempting. Obviously, they eliminate the cooperative payoff in those markets. However, firms gain a benefit from the reduction of efficiency loss. This is because they do not have to worry about the more tempting deviations in markets 1 to k by giving up collusion in the markets. Therefore, the loss attributed to imperfect monitoring comes from the market $k+1$. If the gain exceeds the loss, firms are willing to discard collusion in symmetric equilibrium. Furthermore, continuing the comparison between the gain and the loss from discarding collusion in more tempting markets, they can find *optimal contact size*, $M^* = \{k^*, k^*+1, \cdots, m\}$, where:

$$k^* = \arg\max_{k'} \sum_{k=k'}^{m} \pi^k - \frac{g^{k'}}{l^{k'}-1}.$$

The optimal value can be achieved by the following generalized trigger strategy. In C phase, firms take C in the markets of M^* and take D in markets of $M \setminus M^*$. Start from C phase. If signal B is realized in all the markets of M^*, then the phase transits to D phase with some probability.

3.3 Stability of collusive agreement

We investigate how the lower bound of the discount factor for the optimal trigger strategy responds to changes in market structures. By Proposition 1, we can derive the lower bound of the discount factor as follows.

$$\begin{aligned}\underline{\delta} &= \frac{g^1}{g^1 + (p_2^1 - p_1^1)\Pi_{k'=2}^m(1 - p_2^{k'})v^*} \\ &= \frac{g^1}{g^1 + (p_2^1 - p_1^1)\Pi_{k'=2}^m(1 - p_2^{k'})\left[\sum_{k=1}^m \pi^k - \frac{g^1}{l^1-1}\right]}.\end{aligned} \tag{7}$$

The parameters that constitute the lower bound of the discount factor are classified into three parts: cooperative payoffs, the parameters that determine the degree of temptation to deviate in market 1, and probability of receiving a good signal under (C, C) in a market $k \neq 1$. First, let us consider how the lower bound of the discount factor responds to changes in cooperative payoffs in any market. Differentiate (7) by π^k. Then we have:

$$\frac{\partial \underline{\delta}}{\partial \pi^k} < 0, \tag{8}$$

for $k = 1, 2, \cdots, m$. Therefore, we can confirm that the lower bound of the discount factor decreases as the cooperative payoff in any market increases. This is because the binding incentive constraints become relaxed because collusion becomes more attractive.

Similarly, let us consider how the lower bound of the discount factor responds to changes in the parameters concerning the degree of temptation to deviate in market 1. Then we have:

$$\frac{\partial \underline{\delta}}{\partial g^1} > 0, \ \frac{\partial \underline{\delta}}{\partial p_2^1} < 0, \ \frac{\partial \underline{\delta}}{\partial p_1^1} > 0. \tag{9}$$

By (9), we can also confirm that the lower bound of the discount factor decreases as the degree of temptation in market 1 decreases. This is because the binding incentive constraints become relaxed because the most profitable deviation becomes less attractive.

Finally, let us consider how the lower bound of the discount factor responds to changes in the probability of receiving a good signal under (C, C) in a market $k \neq 1$. Then we have:

$$\frac{\partial \underline{\delta}}{\partial p_2^k} > 0, \tag{10}$$

for $k = 2, 3, \cdots, m$. We can appreciate that if the probability of receiving a good signal under (C, C) in a market $k \neq 1$, p_2^k, increases, the lower bound of the discount factor increases. The reason is as follows. When p_2^k is large, the public signal profiles, including bad signals in markets except k, is realized with relatively higher probability. The signal profile indicates the opponent's deviation in those markets but passes the statistical test whether an opponent deviates, because under the optimal trigger strategy, firms are allowed to trigger *only* when they observe bad signals in *all* markets. Therefore, we can conclude that firms tolerate more bad signals in markets except k prior to triggering as p_2^k becomes large.

These facts have several implications. The first implication is related to what types of multimarket contact makes the collusive agreement stable with respect to the discount factor, given the number of markets they contact. In particular, we can verify how the lower bound of the discount factor to collude is related to the value of collusion. Note that firms can achieve the optimal symmetric equilibrium value without efficiency loss in markets 2 to m. Then, focusing on (8) and (9), we can obtain the following result.

Corollary 1. *Suppose that Assumptions 1, 2, and 3 hold. If $g^1 < \pi^1(l^1 - 1)$, then the lower bound of discount factor, $\underline{\delta}$, decreases as the value of collusion in multimarket contact improves.*

The next implication is related to how the lower bound of the discount factor is affected by diversification. That is, how the lower bound of the discount factor varies with marginal change in the number of markets. For any type of diversification, the common effect of diversification on the lower bound of the discount factor is twofold. By diversification, firms obtain the additional cooperative payoffs. From (8), we can appreciate that the cooperative payoff in the additional market relaxes the incentive constraint to collude. This means that more contacts among firms drive the lower bound of the discount factor to decrease. On the other hand, when firms diversify their business, the optimal trigger strategy requires more tolerance for firms. This is confirmed by the explanation of (10). Then, more contacts among firms drive the lower bound of the discount factor to increase.

To see the effect, we focus on the model in which markets are identical and show that, under some conditions, diversification makes collusive agreement more stable, even though markets are homogeneous. This fact is in contrast with the *irrelevance result* by Bernheim and Whinston (1990). Let us introduce some notations: $\pi \equiv \pi^k$, $g \equiv g^k$, $p_i \equiv p_i^k$ $(i = 1, 2)$, $l \equiv l^k$ for all k. Then, if firms collude in m markets, the lower bound of the discount factor is:

$$\begin{aligned} \underline{\delta}^m &= \frac{g}{g + (p_2 - p_1)(1 - p_2)^{m-1} v^*} \\ &= \frac{g}{g + (p_2 - p_1)(1 - p_2)^{m-1}\Big[m\pi - \frac{g}{l-1}\Big]}. \end{aligned}$$

Now suppose firms diversify into market $m + 1$. Then:

$$\begin{aligned} \underline{\delta}^m > \underline{\delta}^{m+1} \quad &\Leftrightarrow \quad m\pi - \frac{g}{l-1} < (1 - p_2)\big[m\pi - \frac{g}{l-1}\big] + (1 - p_2)\pi \\ &\Leftrightarrow \quad p_2\big[m\pi - \frac{g}{l-1}\big] < (1 - p_2)\pi \qquad (11) \\ &\Leftrightarrow \quad p_2\big[m - \frac{1}{\pi}\big(\frac{g}{l-1}\big)\big] < (1 - p_2). \qquad (12) \end{aligned}$$

The right-hand side of (11) represents the effect of the additional cooperative payoff on the lower bound of the discount factor. The effect is discounted at the rate $1 - p_2$. The left-hand side of (11) embodies the effect of the additional tolerance on the lower bound. When firms diversify into the $m + 1$ market, they tolerate more prior to triggering with probability p_2. Inequality (12) normalizes these effects by π.

Furthermore, from (12), if:

$$m < \frac{1-p_2}{p_2} + \frac{1}{\pi}\left(\frac{g}{l-1}\right) \tag{13}$$

is satisfied, then there is a range of contact size such that diversification is effective not only in improving the value of collusion but also in making collusive agreement more stable. Note that the second term on the left hand side of (13) is less than one because of the sufficient condition that the AMP value is the optimal symmetric equilibrium value. Therefore, diversification is effective not only in improving the value of collusion but also in making collusive agreement more stable only when p_2 is sufficiently small.

For example, consider the case in which $\pi = 2$, $g = 0.5$, $p_2 = 0.4$, and $p_1 = 0.2$. Suppose firms compete with each other in market 1 only. If firms diversify their business into market 2, then the optimal symmetric equilibrium value is 1.25 per market and $\underline{\delta}^{m=2} = \frac{5}{8}$ while the optimal symmetric equilibrium value in market 1 is 0.5 and $\underline{\delta}^{m=1} = \frac{5}{6}$.

The above argument also tells us that there is an upper bound of contact size such that diversification is effective in stabilizing collusive agreement. This means that if firms contact in sufficiently many markets, then the lower bound must increase. This argument implies that, when we interpret the discount factor as the probability of continuing the game, highly diversified firms are limited to large enterprises that are thought of as a "going concern."

4 Exploration of Optimal Equilibrium

In the previous section, we showed the highest value of collusion among generalized trigger strategy equilibria. Although this seems to be the restriction to symmetric pure strategies, asymmetric mixed strategies are easily incorporated into the analysis in light of Proposition 1.[11]

First, let us consider utilizing randomization. Although cooperative payoffs certainly decrease, randomization may improve the value of collusion by alleviating efficiency loss. By Proposition 1, we can verify that efficiency loss is affected by detectability of *the most tempting deviations*.[12] If randomization improves the detectability of those deviations, randomization might be effective in improving the value of collusion.

For simplicity, let us consider the situation in which the market with the most tempting deviation is unique. To confirm the argument, consider the AMP formula of firm i when firms randomize between actions C and D in market 1, and both firms take C in the other markets. Let $\alpha^1 = (\alpha^1_1, \alpha^1_2)$ be a mixed action profile of market 1 in C phase, where $\alpha^1_i = \eta^1_i C + (1-\eta^1_i)D$. Then, the equilibrium payoff v_i satisfies:

$$v_i = (1-\delta)\Big[u^1_i(C,\alpha^1_j) + \sum_{k\neq 1}\pi^k\Big] + \delta\Big[1 - P^1(B|C,\alpha^1_j)\prod_{k\neq 1}P^k(B|C,C)\rho_{(B\cdots B)}\Big]v_i,$$

$$v_i \geq (1-\delta)\Big[u^1_i(D,\alpha^1_j) + \sum_{k\neq 1}\pi^k\Big] + \delta\Big[1 - P^1(B|D,\alpha^1_j)\prod_{k\neq 1}P^k(B|C,C)\rho_{(B\cdots B)}\Big]v_i.$$

If α^1_i mixes C and D, the value function and incentive condition must provide an equal payoff so that the incentive condition is satisfied with equality. Therefore, we can

[11] Kandori and Obara (2006) derive the AMP formula in a version of mixed strategy.

[12] Note that we consider the model including the case in which the degree of temptation to deviate is the same across the markets.

obtain:

$$\begin{aligned} v_i &= u_i^1(C,\alpha_j^1) + \sum_{k\neq 1}\pi^k - \frac{u_i^1(D,\alpha_j^1)-u_i^1(C,\alpha_j^1)}{\frac{P^1(B|D,\alpha_j^1)}{P^1(B|C,\alpha_j^1)}-1} \\ &= \sum_{k\neq 1}\pi^k + \eta_j^1\Big(\pi^1 - \frac{g^1}{\frac{\eta_j^1(1-p_1^1)+(1-\eta_j^1)(1-p_0^1)}{\eta_j^1(1-p_2^1)+(1-\eta_j^1)(1-p_1^1)}-1}\Big) \\ &\quad -(1-\eta_j^1)b^1\Big(1+\frac{1}{\frac{\eta_j^1(1-p_1^1)+(1-\eta_j^1)(1-p_0^1)}{\eta_j^1(1-p_2^1)+(1-\eta_j^1)(1-p_1^1)}-1}\Big). \end{aligned}$$

If $\frac{1-p_1^1}{1-p_2^1} < \frac{1-p_0^1}{1-p_1^1}$, randomization is effective in improving the detactability of the most tempting deviation. Therefore, taking D with small probability might be effective in improving the value of collusion. On the other hand, when $\frac{1-p_1^1}{1-p_2^1} \geq \frac{1-p_0^1}{1-p_1^1}$, randomization *never* improves the value of collusion.[13] Therefore, to show that the value derived in the previous section is optimal among any perfect public equilibria, we require the following condition.

Assumption 4. *Let $K = \{k \mid \frac{g^k}{l^k-1} \geq \frac{g^{k'}}{l^{k'}-1}$ for all $k' \neq k\}$ be the set of markets with the most tempting deviations. For all $k \in K$, $\frac{1-p_1^k}{1-p_2^k} \geq \frac{1-p_0^k}{1-p_1^k}$.*

Assumption 4 says that a unilateral deviation is the most detectable when all firms take "cooperation." Assumption 4 is derived by the complementarity condition that one firm's cooperation has greater effects on the marginal probability of receiving a good signal when the opponent cooperates than when he defects. That is, for all k:

$$p_2^k - p_1^k \geq p_1^k - p_0^k.$$

Next, to get optimal equilibrium payoffs, we also need to take into account cases where the optimal strategy pair is asymmetric. Consider a mixed action profile $\alpha^k = (\alpha_1^k, \alpha_2^k)$ in market k. Let us define $\omega_i(\alpha^k) \in \{G, B\}$ by $P^k(\omega_i(\alpha^k)|D,\alpha_j^k) > P^k(\omega_i(\alpha^k)|C,\alpha_j^k)$. The signal outcome $\omega_i(\alpha)$ represents the one that detects firm i's defection at α^k. If there exists $(\omega_1(\alpha^k), \omega_2(\alpha^k))$ such that $\omega_1(\alpha^k) \neq \omega_2(\alpha^k)$, firm i transfers its continuation payoffs to firm j, when $\omega_i(\alpha^k)$ is realized in market k. However, in our model, because of the failure of statistical distinguishability of defection, that is, $\omega_1(\alpha^k) = \omega(\alpha^k) = B$, we cannot attribute one bad signal to a particular firm's defection. Therefore, to implement a cooperative action profile, mutual punishment is required when a bad signal is realized. Therefore, if asymmetric strategies improve the efficiency, firms must transfer their continuation payoffs to the opponents only by alternating between (D, C) and (C, D) in a market. That is, if $2v^* \leq 2\sum_{k'\neq k}\pi^{k'} + \pi^k + g^k - b^k$ for some k, asymmetric strategy is effective in improving the value of collusion when δ is close to 1. Otherwise, asymmetric strategy *never* improves the value of collusion. To show that asymmetric strategy is not optimal, we introduce the following assumption.

Assumption 5. *For all k:*

$$\pi^k - \frac{2g^1}{l^1-1} > g^k - b^k. \tag{14}$$

[13] This argument is valid for two-player games. For a general discussion about randomization, see the Appendix.

Then, we show in the following proposition the optimality of the AMP value under Assumptions 4 and 5.[14]

Proposition 2. *Suppose that Assumptions 1, 2, and 3 are satisfied. Suppose also that Assumptions 4 and 5 hold.*

If $g^1 < (l^1 - 1)\pi^1$, then for any $\boldsymbol{v} = (v_1, v_2) \neq (v^*, v^*)$ such that:

$$v_1 + v_2 \geq 2v^*,$$

$\boldsymbol{v}$ is not an equilibrium payoff pair under any $\delta \in (0, 1)$.

Proof. Let us fix δ arbitrarily. Suppose that there exists an equilibrium payoff vector $\hat{\boldsymbol{v}} = (\hat{v}_1, \hat{v}_2)$ that satisfies $\hat{v}_1 + \hat{v}_2 \geq 2v^*$ and $\hat{\boldsymbol{v}} \neq \boldsymbol{v^*}$.

Let $\boldsymbol{a^{*k}} = (C, C)$ be the most efficient action profile in market k. Let $\boldsymbol{a^*} = (a^{*1}, a^{*2}, \cdots, a^{*m})$. Let Z be the convex hull of $\{(v^*, v^*)\} \cup \{(u_1(\boldsymbol{a}), u_2(\boldsymbol{a}))|\boldsymbol{a} \neq \boldsymbol{a^*}\}$. Because of Assumption 5, (v^*, v^*) is the unique outcome of $\arg\max_{\boldsymbol{z} \in Z} z_1 + z_2$. Therefore, we have $\hat{\boldsymbol{v}} \notin Z$. Therefore, the separating hyperplane theorem applies. Furthermore, because $(1, 1)$ weakly separates $\hat{\boldsymbol{v}}$ from Z, there exists a strictly positive vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \gg \boldsymbol{0}$ such that:

$$\lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2 > \lambda_1 v^* + \lambda_2 v^*. \tag{15}$$

Without loss of generality, we can assume that $\hat{\boldsymbol{v}}$ maximizes the inner product with $\boldsymbol{\lambda}$ over all equilibrium payoff pairs. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2})$ be the mixed action profile played in the initial period under this equilibrium, where:

$$\begin{aligned} &\boldsymbol{\alpha_i} = (\alpha_i^1, \cdots, \alpha_i^m), \\ &\alpha_i^k = \eta_i^k C + (1 - \eta_i^k) D \text{ for } k = 1, \cdots, m. \end{aligned}$$

Because $\hat{\boldsymbol{v}}$ maximizes the inner product, we have:

$$\lambda_1 u_1(\boldsymbol{\alpha}) + \lambda_2 u_2(\boldsymbol{\alpha}) \geq \lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2 > \lambda_1 u_1(\boldsymbol{a}) + \lambda_2 u_2(\boldsymbol{a}) \tag{16}$$

for any $\boldsymbol{a} \neq \boldsymbol{a^*}$, where the second inequality follows from (15). By the second inequality in (16), $\eta_i^k > 0$ for each i in any market k.

Next, let $\boldsymbol{f_i}(\boldsymbol{\omega}) = (f_i^1(\boldsymbol{\omega}), \cdots, f_i^m(\boldsymbol{\omega}))$ be the continuation payoffs for player i from market 1 to m, when a signal profile $\boldsymbol{\omega}$ is realized in the initial period. Let $\boldsymbol{f}(\boldsymbol{\omega}) = (\boldsymbol{f_1}(\boldsymbol{\omega}), \boldsymbol{f_2}(\boldsymbol{\omega}))$. Because $\hat{\boldsymbol{v}}$ is an equilibrium payoff vector, we have the following value equations and incentive conditions.

$$\begin{aligned} \hat{v}_i &= (1 - \delta) u_i(\boldsymbol{\alpha_i}, \boldsymbol{\alpha_j}) + \delta \sum_{\boldsymbol{\omega}} P(\boldsymbol{\omega}|\boldsymbol{\alpha}) \boldsymbol{f_i}(\boldsymbol{\omega}), \\ \hat{v}_i &\geq (1 - \delta) u_i(\boldsymbol{a_i'}, \boldsymbol{\alpha_j}) + \delta \sum_{\boldsymbol{\omega}} P(\boldsymbol{\omega}|\boldsymbol{a_i'}, \boldsymbol{\alpha_j}) \boldsymbol{f_i}(\boldsymbol{\omega}) \ \forall \boldsymbol{a_i'} \end{aligned}$$

for $i = 1, 2$. If we denote $P^k(\omega^k|\alpha^k) W_i(\omega^k|\boldsymbol{\alpha^{-k}}) \equiv \sum_{\boldsymbol{\omega^{-k}}} P(\omega^k, \boldsymbol{\omega^{-k}}|\boldsymbol{\alpha}) \boldsymbol{f_i}(\omega^k, \boldsymbol{\omega^{-k}})$, where $\boldsymbol{\alpha^{-k}} = (\alpha^1, \cdots, \alpha^{k-1}, \alpha^{k+1} \cdots, \alpha^m)$ and $\boldsymbol{\omega^{-k}} = (\omega^1, \cdots, \omega^{k-1}, \omega^{k+1}, \cdots, \omega^m)$, combined with the fact $\eta_i^k > 0$ for each i in any market k, the above equations imply:

$$\hat{v}_i = (1 - \delta)\{u_i^k(C, \alpha_j^k) + \sum_{\kappa \neq k} u_i^\kappa(\alpha_i^\kappa, \alpha_j^\kappa)\} + \delta \sum_{\omega^k} P^k(\omega^k|C, \alpha_j^k) W_i(\omega^k|\boldsymbol{\alpha^{-k}}), \tag{17}$$

[14]The proof of Proposition 2 is an application of the technique developed by Kobayashi, Ohta and Sekiguchi (2007) to the multimarket contact situation.

$$\hat{v}_i \geq (1-\delta)\{u_i^k(D,\alpha_j^k) + \sum_{\kappa \neq k} u_i^\kappa(\alpha_i^\kappa, \alpha_j^\kappa)\} + \delta \sum_{\omega^k} P^k(\omega^k|D,\alpha_j^k) W_i(\omega^k|\boldsymbol{\alpha}^{-\boldsymbol{k}}) \tag{18}$$

for each i and k.

Now, we focus on $k=1$. Inequality (18) for $k=1$ is equivalent to:

$$\begin{aligned}
&\sum_{\omega^1}\{P^1(\omega^1|C,\alpha_j^1) - P^1(\omega^1|D,\alpha_j^1)\} W_i(\omega^1|\boldsymbol{\alpha}^{-\mathbf{1}}) \\
&\quad \geq \frac{1-\delta}{\delta}\{u_i^1(D,\alpha_j^1) - u_i^1(C,\alpha_j^1)\} \\
&\Leftrightarrow \{P^1(G|C,\alpha_j^1) - P^1(G|D,\alpha_j^1)\}\{W_i(G|\boldsymbol{\alpha}^{-\mathbf{1}}) - W_i(B|\boldsymbol{\alpha}^{-\mathbf{1}})\} \\
&\quad \geq \frac{1-\delta}{\delta}\{u_i^1(D,\alpha_j^1) - u_i^1(C,\alpha_j^1)\} \\
&\Leftrightarrow W_i(G|\boldsymbol{\alpha}^{-\mathbf{1}}) - W_i(B|\boldsymbol{\alpha}^{-\mathbf{1}}) \geq \left(\frac{1-\delta}{\delta}\right)\left[\frac{u_i^1(D,\alpha_j^1) - u_i^1(C,\alpha_j^1)}{P^1(G|C,\alpha_j^1) - P^1(G|D,\alpha_j^1)}\right]. \qquad (19)
\end{aligned}$$

By rearranging the value equation (17) for $k=1$, we can get:

$$\begin{aligned}
\hat{v}_i &= (1-\delta)\{u_i^1(C,\alpha_j^1) + \sum_{\kappa\neq 1} u_i^\kappa(\alpha_i^\kappa,\alpha_j^\kappa)\} \\
&+ \delta W_i(G|\boldsymbol{\alpha}^{-\mathbf{1}}) - \delta P^1(B|C,\alpha_j^1)\{W_i^1(G|\boldsymbol{\alpha}^{-\mathbf{1}}) - W_i(B|\boldsymbol{\alpha}^{-\mathbf{1}})\}.
\end{aligned}$$

From (19), we have:

$$\begin{aligned}
&\hat{v}_i \leq (1-\delta)\{u_i^1(C,\alpha_j^1) + \sum_{\kappa\neq 1} u_i^\kappa(\alpha_i^\kappa,\alpha_j^\kappa)\} \\
&+ \delta W_i(G|\boldsymbol{\alpha}^{-\mathbf{1}}) - (1-\delta)P^1(B|C,\alpha_j^1)\left[\frac{u_i^1(D,\alpha_j^1) - u_i^1(C,\alpha_j^1)}{P^1(G|C,\alpha_j^1) - P^1(G|D,\alpha_j^1)}\right].
\end{aligned} \tag{20}$$

If we apply the same operations for all i, using:

$$\begin{aligned}
&\lambda_1\hat{v}_1 + \lambda_2\hat{v}_2 \geq \lambda_1 W_1(G|\boldsymbol{\alpha}^{-\mathbf{1}}) + \lambda_2 W_2(G|\boldsymbol{\alpha}^{-\mathbf{1}}), \\
&\sum_{\kappa\neq 1}\pi^\kappa \geq \sum_{\kappa\neq 1} u_i^\kappa(\alpha_i^\kappa,\alpha_j^\kappa) \text{ for each } i,
\end{aligned}$$

we have:

$$\begin{aligned}
\sum_{i=1}^2 \lambda_i\hat{v}_i &\leq \sum_{i=1}^2 \lambda_i\left[u_i^1(C,\alpha_j^1) + \sum_{\kappa\neq 1}\pi^\kappa - \frac{u_i^1(D,\alpha_j^1) - u_i^1(C,\alpha_j^1)}{\frac{P^1(B|D,\alpha_j^1)}{P^1(B|C,\alpha_j^1)} - 1}\right] \\
&= \sum_{i=1}^2 \lambda_i\left[\eta_i^1(\pi^1 - \frac{g^1}{\frac{P^1(B|D,\alpha_j)}{P^1(B|C,\alpha_j)} - 1}) - (1-\eta_i^1)b^1(1 + \frac{1}{\frac{P^1(B|D,\alpha_j)}{P^1(B|C,\alpha_j)} - 1}) + \sum_{\kappa\neq 1}\pi^\kappa\right] \\
&\leq \lambda_1(v^{*1} + \sum_{\kappa\neq 1}\pi^\kappa) + \lambda_2(v^{*1} + \sum_{\kappa\neq 1}\pi^\kappa) \\
&= \lambda_1 v^* + \lambda_2 v^*.
\end{aligned}$$

The last inequality holds by Assumption 4. This contradicts Inequality (15). ∎

Proposition 2 states that there exists an upper bound on the sum of equilibrium payoffs (namely, the efficiency level of an equilibrium) that applies to any degree of patience of the players. This "uniform inefficiency" result stems from the same logic as the classic result by Radner, Myerson and Maskin (1986).

Note that the private strategy equilibrium studied by Kandori and Obara (2006) does not work under Assumption 4. A private strategy specifies a current action conditional not only on the public history but also on the private history. When firms adopt a mixed private strategy, a combination of one's own actions in the past and the history of the public signal may contain more information than just a history of the public signal about the opponent's action. However, because under Assumption 4, taking noncooperative action with positive probability simply worsens the statistical detectability of deviations, the combination of the actions and the signals does not contain more information. Although whether other types of private strategy equilibria improve the AMP value is an open question, we conjecture that the AMP value is optimal among any sequential equilibria and the efficiency loss is truly inevitable under Assumption 4 and 5.

5 Concluding Remarks

We have investigated the optimal equilibrium value in multimarket contact with imperfect public monitoring. Specifically, we showed that patient firms can collude without efficiency loss in markets except the market with the most tempting deviation. We can appreciate that by sampling an opponent's behavior from multiple markets, firms can conduct a more accurate test than in the case of a single market situation, and therefore alleviate the efficiency loss.

The results in this paper depend on the assumption that the signals are independently distributed among markets. We believe that in considering the case of correlated signals among markets, the optimal strategy will be the same as in the independent signal case. This is because firms would like to use punishment that is minimal, but that is also severe enough to maintain firms' incentive to collude. This issue remains to be examined in future research.

Appendix

In this appendix, we extend the AMP formula and the optimality result in sections 3 and 4 to the n-firm model. Let $N = \{1, 2, \cdots, n\}$ be the set of firms. Each firm selects an action a_i^k from $\{C, D\}$ for each market k. Given an action profile $\boldsymbol{a} = (a_1^1, a_1^2 \cdots, a_1^m; a_2^1, a_2^2, \cdots, a_2^m; \cdots; a_n^1, a_n^2 \cdots, a_n^m) \in A$, firms observe a vector of public signals, $\boldsymbol{\omega} = (\omega^1, \omega^2, \cdots, \omega^m)$. Each ω^k takes a value, G or B according to the probability distribution $p_{n'}^k = P^k(G|(\underbrace{C, \cdots, C}_{n'}, \underbrace{D, \cdots, D}_{n-n'}))$. The following assumption is a natural extension of Assumption 2 to the n-firm environment.

Assumption 6. *For any k, $p_{n'}^k$ is monotonically increasing in n'. That is, $p_n^k > p_{n-1}^k > \cdots > p_0^k$.*

Note that because n firms are symmetric in each market, the expected profit for firm

i depends on firm i's action and the number of firms other than i taking "cooperation." For simplicity, we denote an expected payoff for firm i from market k by $u_i^k(a_i^k, \boldsymbol{C^{n'}})$, which stands for firm i's payoff when firm i takes a_i^k and $n'(= 0, \cdots, n-1)$ firms other than i take C in market k. Because of symmetry, $u_i^k(C, \boldsymbol{C^{n'}}) = u_j^k(C, \boldsymbol{C^{n'}}) \equiv u^k(C, \boldsymbol{C^{n'}})$ and $u_i^k(D, \boldsymbol{C^{n'}}) = u_j^k(D, \boldsymbol{C^{n'}}) \equiv u^k(D, \boldsymbol{C^{n'}})$ for all n'.

To make the stage game be an n-firm version of Prisoners' Dilemma, we make the following assumption.

Assumption 7. *For all k:*

$$
\begin{aligned}
&(C, C, \cdots, C) = \arg\max_{\boldsymbol{a^k}} \sum_{i=1}^{n} u_i^k(\boldsymbol{a^k}),\\
&u^k(D, \boldsymbol{C^{n'}}) > u^k(C, \boldsymbol{C^{n'}}) \text{ for any } n' = 0, 1, \cdots n-1\\
&u^k(D, D, \cdots, D) = 0.
\end{aligned}
$$

Let $l_{n'}^k \equiv \frac{1-p_{n'-1}^k}{1-p_{n'}^k}$ for $n' = 1, 2, \cdots, n$ and $g^k = u^k(D, \boldsymbol{C^{n-1}}) - u^k(C, \boldsymbol{C^{n-1}})$. If we, without loss of generality, permutate the index of the markets as $\frac{g^1}{l_n^1 - 1} \geq \frac{g^2}{l_n^2 - 1} \geq \cdots \geq \frac{g^m}{l_n^m - 1}$, we obtain the n-firm version of the AMP formula by applying the same proof of Proposition 1.

Proposition 3. *Suppose that Assumptions 1, 6, and 7 hold. If $g^1 < u^1(C, \boldsymbol{C^{n-1}})(l_n^1 - 1)$, then there exists $\underline{\delta} \in (0,1)$ such that for any $\delta \geq \underline{\delta}$ the optimal symmetric equilibrium value is:*

$$
v^* = \sum_{k=1}^{m} u^k(C, \boldsymbol{C^{n-1}}) - \frac{g^1}{l_n^1 - 1}. \tag{21}
$$

Now we investigate the optimality of the AMP value (21) in the n-firm case. Remember that, to obtain the optimality result, we must show that randomization and asymmetric strategy do not improve the efficiency. First, let us consider the possibility of randomization. As in the two-firm setting, we assume increasing returns of detectability to scale in order to state that randomization does not improve the detectability of deviations.

Assumption 8. *For all k, $l_n^k \geq l_{n-1}^k \geq \cdots \geq l_1^k$.*

In the n-firm case, Assumption 8 is not sufficient to say randomization does not improve the efficiency. Consider the AMP formula of firm i in market 1 when the other firms randomize between actions C and D. Let us denote the mixed action by $\alpha_j^1 = \eta_j^1 C + (1 - \eta_j^1) D$. Then, the formula can be expressed as:

$$
\begin{aligned}
&u_i^1(C, \boldsymbol{\alpha_{-i}^1}) - \frac{u_i^1(D, \boldsymbol{\alpha_{-i}^1}) - u_i^1(C, \boldsymbol{\alpha_{-i}^1})}{l^1(C, \boldsymbol{\alpha_{-i}^1}) - 1}\\
&= u_i^1(C, \alpha_j^1, \boldsymbol{\alpha_{-i,j}^1}) - \frac{u_i^1(D, \alpha_j^1, \boldsymbol{\alpha_{-i,j}^1}) - u_i^1(C, \alpha_j^1, \boldsymbol{\alpha_{-i,j}^1})}{l^1(C, \alpha_j^1, \boldsymbol{\alpha_{-i,j}^1}) - 1},
\end{aligned}
$$

where $l^1(C, \boldsymbol{\alpha_{-i}}) \equiv \frac{P^1(B|D,\boldsymbol{\alpha_{-i}})}{P^1(B|C,\boldsymbol{\alpha_{-i}})}$. Furthermore, it can be transformed into:

$$\begin{aligned} &\eta_j^1\left[u_i^1(C, C, \boldsymbol{\alpha^1_{-i,j}}) - \frac{u_i^1(D, C, \boldsymbol{\alpha^1_{-i,j}}) - u_i^1(C, C, \boldsymbol{\alpha^1_{-i,j}})}{l^1(C, \alpha_j^1, \boldsymbol{\alpha_{-i,j}}) - 1}\right] \\ &+ (1-\eta_j^1)\left[u_i^1(C, D, \boldsymbol{\alpha^1_{-i,j}}) - \frac{u_i^1(D, D, \boldsymbol{\alpha^1_{-i,j}}) - u_i^1(C, D, \boldsymbol{\alpha^1_{-i,j}})}{l^1(C, \alpha_j^1, \boldsymbol{\alpha_{-i,j}}) - 1}\right]. \end{aligned} \tag{22}$$

Note that in the two-firm environment, Assumption 8 is sufficient to show randomization does not improve the efficiency. Because the second brace of (22) is always negative in the two-firm environment, the first brace is always larger than the second. Therefore, efficiency does not improve if firms put positive probability on D. However, in the n-firm environment, it is not always possible to determine whether the first brace is larger than the second. Therefore, we introduce the following assumption.

Assumption 9. *For any i, $j(\neq i)$,*

$$u_i^1(C, C, \boldsymbol{a^1_{-i,j}}) - u_i^1(C, D, \boldsymbol{a^1_{-i,j}}) \geq u_i^1(D, C, \boldsymbol{a^1_{-i,j}}) - u_i^1(D, D, \boldsymbol{a^1_{-i,j}}) \geq 0$$

for all $\boldsymbol{a^1_{-i,j}}$.

The first inequality of Assumption 9 implies that deviation gain (numerators of efficiency loss in each brace of (22)) is nonincreasing in the number of cooperative firms. The first inequality is satisfied when cooperative action has complementarity relation for each firm. The second inequality means that $u_i^k(a_i^k, \boldsymbol{C^{n'}})$ is nondecreasing in n' for $a_i^k = C, D$, which is a natural assumption.

Next we consider the possibility of asymmetric strategies. Showing that asymmetric strategies do not improve efficiency is the same as in the case of two firms. That is, to achieve optimal asymmetric equilibrium, firms can transfer their continuation payoffs to their opponents by changing the action profile among $(D, \boldsymbol{C^{n'}})$ and $(C, \boldsymbol{C^{n'-1}})$. The following assumption is a generalization of Assumption 5.

Assumption 10. *For all k and $n' = 1, 2, \cdots, n-1$:*

$$n\left[u^k(C, \boldsymbol{C^{n-1}}) - \frac{g^1}{l_n^1 - 1}\right] > (n-n')u^k(D, \boldsymbol{C^{n'}}) + n'u^k(C, \boldsymbol{C^{n'-1}}). \tag{23}$$

With all those assumptions, we obtain the optimality result in the n-firm case by applying the same proof of Proposition 2.

Proposition 4. *Suppose that Assumptions 1, 6, and 7 are satisfied. Suppose also that Assumptions 8, 9, and 10 hold.*

If $g^1 < u^1(C, \boldsymbol{C^{n-1}})(l_n^1 - 1)$, then for any $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \neq (v^*, v^*, \cdots, v^*)$ such that:

$$\sum_{i=1}^{n} v_i \geq nv^*,$$

$\boldsymbol{v}$ is not an equilibrium payoff pair under any $\delta \in (0, 1)$.

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