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Abstract

We investigate the possibility of a Pareto-improving income redistribution when public goods are voluntarily provided. Cornes and Sandler (2000, *German Economic Review* 1, pp. 169-186) derived a condition to find a Pareto-improving redistribution in the presence of a standard public good. We examine this possibility in the presence of standard and weaker-link public goods. Further, we present a case in which there exists no Pareto-improving redistribution with the condition derived by Cornes and Sandler (2000) being satisfied.

Keywords: Public Goods, Pareto-improving Redistribution, Income disparity
JEL classification: D31, F35, H41

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1 Introduction

This note investigates the welfare effects of income redistribution in the presence of voluntarily provided public goods. A considerable number of studies have been conducted on this issue. It has been recognized that the effect depends on whether the income redistribution takes place among the contributors of public goods. Any redistribution among the contributors does not affect the equilibrium levels of the consumption of private and public goods; it leaves individual welfare unaltered (Warr, 1983; Bergstrom et al., 1986). On the other hand, the effect of redistribution from noncontributors to contributors on the welfare of individuals is ambiguous. Such redistribution increases the level of the equilibrium provision of the public goods (Bergstrom et al., 1986). Redistribution will benefit not only the recipient but also the donor if the benefit derived by the donor from an increase in the provision of public goods exceeds the cost incurred by the same from a loss in income.

Several authors have previously explored this possibility. Itaya et al. (1997) showed that a transfer of income from a marginal contributor, who is indifferent between contribution and noncontribution, to the contributors will increase social welfare. Boadway and Hayashi (1999) showed that in an economy where individuals have identical preferences, a transfer of income from noncontributors to contributors may be Pareto improving. Cornes and Sandler (2000) investigated the circumstances under which a Pareto-improving transfer is possible in the presence of voluntarily provided public goods. Further, they derived a condition to find a Pareto-improving redistribution from noncontributors to contributors.

These results cast doubt upon the effectiveness of traditional international aids. The provision of many international public goods relies mainly on the contributions from developed countries (e.g., financial contributions to international organizations). Thus, a reduction in international aid would benefit recipient countries if it resulted in an increase in the contribution from donor countries to international public goods and if the benefits derived by the recipient countries from the increase were sufficiently large.

In recent years, new types of international public goods are becoming increasingly important. For example, the containment or eradication of emerging and re-emerging infectious diseases (e.g., bird flue, malaria, and HIV/AIDS) are regarded, more than ever, as international public goods.[1] These new public goods differ from standard public goods; the total supply of the latter is equal to the sum of the contributions made. However, countries are unable to derive benefits from the former public good until every country contributes a sufficient amount to the public good. For example, for countries to derive benefits from the eradication of an infectious disease, this disease must be eradicated from every country across the globe.

Such new public goods have been formally investigated since the study conducted by Hirshleifer (1983). They are characterized by the manner in which contributions are aggregated for their total provision. One of these public goods is a weaker-link public good; it was initially defined by Cornes (1993) as a public good for which the total provisions equal the geometric mean of the contributions made for it. A marginal increase in the smallest contribution to a weaker-link public good causes the largest increase in its total supply; that in the second-smallest contribution causes the second-largest increase, and so forth. Examples of weaker-link public goods include pest eradication, protection of computer networks against viruses, fiscal and monetary disciplines of a monetary union, and international dispersion of pollutants (Arce and Sandler, 2001).

[1] For health as a global public good, see Chen et al. (1999), Sandler and Arce (2002), Arhin-Tenkorang and Conceição (2003), and Smith et al. (2003).

Cornes (1993) investigated the manner in which a transfer of income between individuals affects their welfare in the presence of a voluntarily provided weaker-link public good. He also showed that when income disparity is sufficiently large, a transfer of income from high-income to low-income individuals is Pareto improving. The question now arises: what type of redistribution is Pareto improving when there are not only standard public goods but also weaker-link public goods.

We develop a simple model where standard and weaker-link public goods are voluntarily provided. In the following framework, we investigate the conditions for a Pareto-improving redistribution and present a case in which although the condition derived by Cornes and Sandler (2000) is satisfied, Pareto-improving redistribution is not achieved.

This paper is an extension of Nakagawa (2003) that presented a case in which although the condition derived by Cornes and Sandler (2000) is satisfied, there is no Pareto-improving redistribution in the form of a transfer from poor to rich individuals.

When the richest individual contributes to both standard and weaker-link public goods and other individuals contribute only to the weaker-link public goods, the results obtained can be summarized as follows.

1. There is a case in which the condition derived by Cornes and Sandler (2000) is satisfied; however, no Pareto-improving redistribution exists.

2. If the contributor of a standard public good receives a sufficiently small share of the aggregate income, there is a Pareto-improving redistribution from the noncontributors to the contributors of the public good.

3. If one of the poorer individuals receives a sufficiently small share of the aggregate income, there is a Pareto-improving redistribution from the other individuals to this individual.

Although this paper considers a very simplified model, it suggests that if public goods are voluntarily provided, the effects of redistribution depend on the configuration of these public goods.

This paper proceeds as follows. In Section 2, we review the condition for a Pareto-improving redistribution derived by Cornes and Sandler (2000). In Section 3, we construct our model consisting of standard and weaker-link public goods. In Section 4, we derive the equilibrium of our model. In Section 5, we examine the welfare effects of redistribution. In Section 6, we investigate when a Pareto-improving redistribution exists and when it does not. We then present a case in which there is no Pareto-improving redistribution even though the condition derived by Cornes and Sandler (2000) is satisfied.

2 The condition derived by Cornes and Sandler

In this section, we review the condition for a Pareto-improving redistribution derived by Cornes and Sandler (2000). Let us consider the example of an economy with a population equal to N where each individual consumes a private good and contributes to a standard public good. Let the private good consumption of an individual i be denoted by y_i and the total supply of the standard public good be denoted by Q. Therefore, individual i's utility function is given by

$$u_i\left(y_i, Q\right),$$

which is strictly increasing, strictly quasi-concave, and twice continuously differentiable for all arguments. Individual i's budget constraint is given by

$$m_i = y_i + q_i,$$

where m_i and q_i are his/her income and contribution to the standard public good, respectively. The total supply of the standard pubic good is the sum of the contributions made by all the N people, $Q = \sum_i q_i$. Further, let us denote individual i's indirect utility function by $V_i(m_i, Q)$.

At initial equilibrium, let the aggregate income received by the contributors be denoted by M_c. Further, let the aggregate contribution function that gives the equilibrium supply of the public good as a function of the aggregate income received by the contributors be denoted by $F_c(M_c)$. Representing the set of noncontributors by $\bar{C}$, proposition 1 of Cornes and Sandler (2000) can be stated as follows:

Proposition 1 (Cornes and Sandler). *Suppose that any $(dm_1, dm_2, \ldots, dm_N)$ such that $\sum_{i=1}^{N} dm_i \leq 0$ is feasible. If, in the neighborhood of the initial equilibrium,*

$$\left(\sum_{i\in\bar{C}} \frac{\partial V_i/\partial Q}{\partial V_i/\partial m_i}\right)\left(\frac{dF_c}{dM_c}\right) > 1 \tag{1}$$

then we can find a locally Pareto-improving redistribution.

The LHS of equation (1) is the sum of the marginal benefits accrued by the noncontributors and the RHS is the sum of the marginal cost incurred by them. If the former is greater than the latter, there is a Pareto-improving redistribution from noncontributors to contributors.

In the rest of this paper, we investigate the welfare effects of redistribution in the presence of standard and weaker-link public goods. As Cornes and Sandler (2000) mentioned, if all goods are normal goods and if there is only one noncontributor, the condition derived by them is not satisfied. Hence, we consider a three-person economy and concentrate on an equilibrium where two individuals do not contribute to the standard public goods.

3 The model

Consider an economy with three individuals in which standard and weaker-link public goods are voluntarily provided. Individuals are indexed by i $(i = 1, 2, 3)$. Each individual consumes a private good, a standard public good, and a weaker-link public good. Let the private good consumption of individual i be denoted by y_i, and the total supply of the weaker-link and standard public goods by X and Q, respectively. For simplicity, we assume that individual i's utility function has an identical Cobb-Douglas form:

$$u_i(y_i, X, Q) = y_i^{\alpha} X^{\beta} Q^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0,$$

where α, β, γ are parameters representing the preferences of individuals for the private, weaker-link public, and standard public goods. Further, we denote individual i's contributions to the weaker-link and standard public goods by x_i and q_i, respectively. Therefore, the total supply of public goods is given by

$$X=\prod_{i=1}^{3}(x_i)^{1/3}, Q=\sum_{i=1}^{3}q_i.$$

By choosing the units of measurement appropriately, we assume that each of the price of private good and the cost of one unit of contribution is equal to unity. Then, individual i's budget constraint is given by

$$m_i=y_i+x_i+q_i,$$

where m_i is the individual's income. Without loss of generality, let $m_1<m_2<m_3$. Let the aggregate income is denoted by M, i.e., $M=\sum_{i=1}^{3}m_i$. We assume that every individual contributes a positive amount to the weaker-link public good and a nonnegative amount to the standard public good, that is, $x_i>0,\ q_i\geq 0$.[2] Further, we suppose that an individual considers the contribution made by other individuals as given. Thus, individual i's utility maximization problem is given by

$$\begin{aligned}\max_{\{y_i,x_i,q_i\}} u_i &= y_i^{\alpha}X^{\beta}Q^{\gamma}\\ \text{s.t. } m_i &= y_i+x_i+q_i,\\ X &= \left(x_i\tilde{X}_i\right)^{1/3}, Q=q_i+\tilde{Q}_i, q_i\geq 0, \text{ and } x_i>0,\end{aligned}$$

where $\tilde{X}_i$ is the product of the contributions made by other individuals to the weaker-link public good, i.e., $\tilde{X}_i=\prod_{j\neq i}x_j$, and $\tilde{Q}_i$ is the sum of the contributions made by other individuals to the standard public good, i.e., $\tilde{Q}=\sum_{j\neq i}q_j$. Solving the utility maximization problem, individual i's contribution functions are

$$x_i\left(m_i,\tilde{X}_i,\tilde{Q}_i\right)=\begin{cases}\frac{(\beta/3)}{\alpha+(\beta/3)}m_i & \text{if } \tilde{Q}_i\geq\frac{\gamma}{\alpha+(\beta/3)}m_i\\ \frac{(\beta/3)}{\alpha+(\beta/3)+\gamma}\left(m_i+\tilde{Q}_i\right) & \text{if } \tilde{Q}_i<\frac{\gamma}{\alpha+(\beta/3)}m_i\end{cases}, \quad (2)$$

$$q_i\left(m_i,\tilde{X}_i,\tilde{Q}_i\right)=\begin{cases}0 & \text{if } \tilde{Q}_i\geq\frac{\gamma}{\alpha+(\beta/3)}m_i\\ \frac{\gamma}{\alpha+(\beta/3)+\gamma}\left\{m_i-\frac{[\alpha+(\beta/3)]}{\gamma}\tilde{Q}_i\right\} & \text{if } \tilde{Q}_i<\frac{\gamma}{\alpha+(\beta/3)}m_i\end{cases}, \quad (3)$$

and his/her indirect utility function is

$$V_i\left(m_i,\tilde{X}_i,\tilde{Q}_i\right)=\begin{cases}\frac{\alpha^{\alpha}(\beta/3)^{\beta/3}}{[\alpha+(\beta/3)]^{\alpha+(\beta/3)}}(m_i)^{\alpha+\frac{\beta}{3}}\left(\tilde{X}_i\right)^{\frac{\beta}{3}}\left(\tilde{Q}_i\right)^{\gamma} & \text{if } \tilde{Q}_i\geq\frac{\gamma}{\alpha+(\beta/3)}m_i\\ \frac{\alpha^{\alpha}(\beta/3)^{\beta/3}\gamma^{\gamma}}{[\alpha+(\beta/3)+\gamma]^{\alpha+(\beta/3)+\gamma}}\left(m_i+\tilde{Q}_i\right)^{\alpha+\frac{\beta}{3}+\gamma}\left(\tilde{X}_i\right)^{\frac{\beta}{3}} & \text{if } \tilde{Q}_i<\frac{\gamma}{\alpha+(\beta/3)}m_i\end{cases}. \quad (4)$$

4 The equilibrium

The equilibrium levels of contributions are obtained as the solution to the following system of equations:

[2] Cornes(1993) implicitly assumed that every individual contributes to the weaker-link public good. The assumption can be justified since when individual i does not contribute to the weaker-link public good, his utility becomes zero.

$$x_i = x_i\left(m_i, \prod_{j\neq i} x_j, \sum_{j\neq i} q_j\right),\ q_i = q_i\left(m_i, \prod_{j\neq i} x_j, \sum_{j\neq i} q_j\right), \text{for } i = 1, 2, 3.$$

Based on Appendix A, an equilibrium where two individuals contribute only to the weaker-link public good can be attained if and only if

$$m_2 \leq \frac{\alpha + (\beta/3)}{\alpha + (\beta/3) + \gamma} m_3. \tag{5}$$

At such an equilibrium, the level of an individual's contribution is given by

$$x_i^N = \frac{(\beta/3)}{\alpha + (\beta/3)} m_i, \qquad q_i^N = 0, \quad \text{for } i = 1, 2; \tag{6}$$

$$x_3^N = \frac{(\beta/3)}{\alpha + (\beta/3) + \gamma} m_3,\ q_3^N = \frac{\gamma}{\alpha + (\beta/3) + \gamma} m_3. \tag{7}$$

Note that the equilibrium level of contribution of an individual depends only on his/her income.

5 The welfare effects of redistribution

Differentiating the indirect utility function of individual $i\,(i = 1, 2)$, who contributes only to the weaker-link public good, with respect to income, we obtain

$$(v_{im})^{-1} du_i = dm_i + v_{ij} x_j' dm_j + \left(v_{i3} x_3' + v_{iQ} Q'\right) dm_3, \tag{8}$$

where

$$v_{im} \equiv \frac{\partial V_i}{\partial m_i},\ v_{ij} \equiv \frac{\left(\partial V_i / \partial \tilde{X}_i\right)}{(\partial V_i / \partial m_i)} \left(\frac{\partial \tilde{X}_i}{\partial x_j}\right),\ v_{iQ} \equiv \frac{\left(\partial V_i / \partial \tilde{Q}_i\right)}{(\partial V_i / \partial m_i)},$$

$$x_i' \equiv \frac{dx_i^N}{dm_i},\ Q' \equiv \frac{d}{dm_3}\left(q_1^N + q_2^N + q_3^N\right),\ j = 1, 2, 3,\ j \neq i.$$

As an illustration, let us imagine an income redistribution from individual 3 to i and j. The first term on the RHS of equation (8) is the direct income effect, that is, an increase in i's income improves his/her welfare. The second term represents the externality effect from individual j through the weaker-link public good, namely, an increase in j's income raises his/her contribution to the weaker-link public good; this in turn improves i's welfare. The third term indicates the externality effect from individual 3 through both types of public goods, i.e., a decrease in individual 3's income reduces the level of his/her contributions to both types of public goods; this in turn reduces i's welfare.

Similarly, differentiating the indirect utility function of individual 3 with respect to income yields

$$(v_{3m})^{-1} du_3 = dm_3 + v_{31} x_1' dm_1 + v_{32} x_2' dm_2, \tag{9}$$

where $v_{3m} \equiv \frac{\partial V_3}{\partial m_3},\ v_{3j} \equiv \frac{\left(\partial V_3/\partial \tilde{X}_3\right)}{(\partial V_3/\partial m_3)}\left(\frac{\partial \tilde{X}_3}{\partial x_j}\right),\ j = 1, 2$. Since individuals 1 and 2 do not contribute to the standard public good, there is no externality effect through the standard public good on the RHS of equation (9).

From equations (4), (6), and (7), we have

$$v_{ij} = \frac{m_i}{m_j};\ v_{iQ} = v_{i3} = (v_{3i})^{-1} = \left[\frac{\alpha + (\beta/3) + \gamma}{\alpha + (\beta/3)}\right] \frac{m_i}{m_3};\ i, j = 1, 2;\ j \neq i. \tag{10}$$

This implies that the externality effect on an individual from another individual through public goods increases with the former's income and decreases with the latter's.

6 The possibility of a Pareto-improving redistribution

In this section, we examine the conditions under which a Pareto-improving redistribution exists. We present a case in which the condition imposed by proposition 1 is satisfied; however, no Pareto-improving redistribution is achieved.

To begin with, let us rewrite the condition imposed by proposition 1 in our model. Since we are concentrating on the equilibrium where individual 3 alone contributes to the standard public good, the condition imposed by proposition 1 can be rewritten as follows:

$$(v_{1Q} + v_{2Q})\, Q' > 1.$$

From equations (7) and (10), we have

$$\frac{\gamma}{\alpha + (\beta/3) + \gamma} > \frac{m_3}{M}. \tag{11}$$

In the remaining section, we show that a Pareto-improving redistribution does not exist with condition (11) being satisfied if we have

$$\frac{m_1}{M} > \frac{(\beta/3)}{\alpha + \beta + \gamma} \text{ and } \frac{\gamma}{\alpha + (\beta/3) + \gamma} > \frac{m_3}{M} > \frac{(\beta/3) + \gamma}{\alpha + \beta + \gamma}.$$

Since conditions (5) and (11) imply $\gamma > \alpha + (\beta/3)$, we assume the relationship

$$\gamma > \alpha + (\beta/3)\,. \tag{12}$$

6.1 Pareto-improving marginal change in income

Let us investigate the conditions under which a Pareto-improving redistribution exists. A marginal change in income can be qualified as a Pareto-improving redistribution if it satisfies the following two conditions:

1. It improves the welfare of at least one individual without decreasing that of the others.

2. The aggregate income after the marginal change is not greater than the initial aggregate income.

A marginal change in income satisfying the first condition is hereafter referred to as a *Pareto-improving marginal change in income*. First, we derive a Pareto-improving marginal change in income. Then, we investigate whether this Pareto-improving marginal change satisfies the second condition. The following reasoning is an application of Tucker's theorem of the alternatives.[3]

[3] For Tucker's theorem of the alternatives, see Mangasarian(1969). For its application in economics, see Myles(1995). His interpretation of this theorem is from the view point of welfare maximiziation, while our

Rewriting equations (8) and (9) in the matrix form yields

$$\mathbf{du} = \mathbf{A}\mathbf{dm}, \tag{13}$$

where

$$\mathbf{du} = \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}, \mathbf{A} = \begin{bmatrix} v_{1m} & v_{1m}v_{12}x_2' & v_{1m}\left(v_{13}x_3' + v_{1Q}Q'\right) \\ v_{2m}v_{21}x_1' & v_{2m} & v_{2m}\left(v_{23}x_3' + v_{2Q}Q'\right) \\ v_{3m}v_{31}x_1' & v_{3m}v_{32}x_2' & v_{3m} \end{bmatrix}, \mathbf{dm} = \begin{pmatrix} dm_1 \\ dm_2 \\ dm_3 \end{pmatrix}.$$

Let Δ be the determinant of the matrix $\mathbf{A}$ divided by $v_{1m}v_{2m}v_{3m}$, that is, $\Delta \equiv |\mathbf{A}| / (v_{1m}v_{2m}v_{3m})$. Then, we have

$$\begin{aligned} \Delta &= 1 - \left(x_1'x_2' + x_2'x_3' + x_3'x_1'\right) + 2x_1'x_2'x_3' + \left(2x_1'x_2' - x_2' - x_1'\right)Q' \\ &= \frac{\alpha^2(\alpha+\beta+\gamma)}{[\alpha+(\beta/3)]^2[\alpha+(\beta/3)+\gamma]} > 0. \end{aligned}$$

Solving equation (13) with respect to $\mathbf{dm}$ yields

$$\mathbf{dm} = \mathbf{A}^{-1}\mathbf{du} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)\,\mathbf{du}, \tag{14}$$

where $\mathbf{a}_i$ $(i = 1, 2, 3)$ is the ith column vector of the inverse matrix of $\mathbf{A}$. These vectors are presented in Appendix B.

Substituting $du_i = 1;\ du_j = du_k = 0;\ i, j, k = 1, 2, 3;\ j, k \neq i$ in equation (14), we observe that the Pareto-improving marginal change in income that improves individual i's welfare by one unit without affecting the other individuals' welfare is given by

$$\mathbf{dm} = \mathbf{a}_i.$$

Turning now to the second condition, the marginal change needs to be accompanied by a nonincrease in the aggregate income, that is, $dM = \sum_i dm_i \leq 0$.

6.2 Can the Pareto-improving marginal change be accomplished by redistribution?

For a Pareto-improving marginal change in income that improves the welfare of individual i without affecting those of others, the second condition can be written as

$$\frac{\partial M}{\partial u_i} \equiv \left.\frac{d(m_1+m_2+m_3)}{du_i}\right|_{du_j=0,\ j\neq i} = \mathbf{a}_i^t\mathbf{b} \leq 0, \tag{15}$$

where $\mathbf{b}^t = (1, 1, 1)$ and the super-script t indicates a transposition of the original matrix. $\partial M/\partial u_i$ is the marginal change in the aggregate income as a result of the Pareto-improving marginal change in income. If $\partial M/\partial u_i$ is nonpositive, the marginal change in income can be accomplished by redistribution; however, if it is positive, the marginal change would require an accompanied growth in the aggregate income, and would not be accomplished by mere redistribution. In other words, $\mathbf{a}_i$ can be regarded as a Pareto-improving redistribution only if $\partial M/\partial u_i$ is nonpositive and vice versa.

First, let us consider the marginal change in income that improves the welfare of

interpretation is from the view point of aggregate income minimization.

individual 3 without affecting those of the others. From equation (15), we have

$$\Delta v_{3m}\frac{\partial M}{\partial u_3} = 1-(v_{1Q}+v_{2Q})\,Q'-(v_{13}+v_{23})\,x_3'+v_{21}x_1'\left(v_{13}x_3'+v_{1Q}Q'\right) +v_{12}x_2'\left(v_{23}x_3'+v_{2Q}Q'\right)-v_{21}x_1'v_{12}x_2'. \quad (16)$$

Let us now analyze the terms appearing on the RHS of equation (16). The direct income effect of a marginal change on individual 3 is represented by the first term. The following terms correspond to the externality effects of the same. If an externality effect benefits individual $i\,(i \neq 3)$, it reduces the aggregate income after the marginal change. Since the marginal change does not alter the welfare of individual i, his/her income is decreased in order to nullify the beneficial externality effect; this means a decrease in the aggregate income. On the contrary, if an externality effect harms individual $i\,(i \neq 3)$, the compensation required to offset the harmful effect would necessitate an increase in his/her income; this means an increase in the aggregate income.

The second term corresponds to the direct externality effect of an increase in the level of contribution made by the third individual to the standard public good q_3. As a result of a growth in individual 3's income, he/she raises the contribution to the standard public good; this benefits individuals 1 and 2. The first two terms in equation (16) correspond with the terms on the LHS of equation (1).

Unlike equation (1), there also exist externality effects through the weaker-link public good in equation (16). The third term in equation (16) corresponds to the direct externality effect of an increase in the level of contribution made by the third individual x_3. As a result of a growth in individual 3's income, he/she raises the contribution to the weaker-link public good; this benefits individuals 1 and 2.

The fourth term in equation (16) corresponds to the indirect externality effect of decrease in the level of contribution made by the first individual x_1. The marginal change reduces individual 1's income by the direct externality effect from the third individual, i.e., $v_{13}x_3' + v_{1Q}Q'$. As a consequence of this decline, individual 1 contributes less to the weaker-link public good; this harms individual 2. Similarly, the fifth term of equation (16) is the indirect externality effect of a decrease in the level of contribution made by the second individual x_2.

The sixth term of equation (16) is the indirect externality effect of an increase in the levels of contribution made by the third individual, i.e., x_3 and q_3 through the level of contribution made by individual i $(i = 1, 2)$. As a result of a growth in the levels of contribution made by the third individual, individual i contributes more to the weaker-link public good; this in turn benefits individual j $(j = 1, 2;\ j \neq i)$.

Substituting equations (6), (7), and (10) in equation (16) yields

$$\Delta v_{3m}\frac{\partial M}{\partial u_3} = M_{3Q} + M_{3X}, \quad (17)$$

$$= \frac{\alpha\,(\alpha+\beta+\gamma)}{[\alpha+(\beta/3)]^2}\left[1-\frac{(\beta/3)+\gamma}{\alpha+\beta+\gamma}\left(\frac{M}{m_3}\right)\right] \leq 0, \quad (18)$$

where

$$M_{3Q} \equiv \frac{[\alpha + (\beta/3) + \gamma]}{[\alpha + (\beta/3)]} \left[1 - \frac{\gamma}{\alpha + (\beta/3) + \gamma} \left(\frac{M}{m_3} \right) \right],$$

$$M_{3X} \equiv \frac{(\beta/3) [\gamma - \alpha + (\beta/3)]}{[\alpha + (\beta/3)]^2} \left[\frac{\gamma - \alpha}{\gamma - \alpha + (\beta/3)} \left(\frac{M}{m_3} \right) - 1 \right].$$

The first term on the RHS of equation (17), i.e., M_{3Q}, is the sum of the first two terms on the RHS of equation (16); this term increases with an increase in the share of individual 3's income, i.e., m_3/M. The second term, M_{3X}, is the sum of the last four terms on the RHS of equation (17); this term decreases with an increase in the share of individual 3's income. If M_{3Q} is negative, then the condition imposed by proposition 1 is satisfied. If $M_{3Q} + M_{3X}$ is nonpositive, then $\mathbf{a}_3$ is regarded as a Pareto-improving redistribution.

(FIGURE 1)

Figure 1 depicts how these terms in equation (17) change with a change in the share of individual 3's income. When the share of individual 3's income is small, M_{3Q} is not greater than zero, but M_{3X} is positive; this implies that $M_{3Q} + M_{3X}$ is greater than M_{3Q}. Thus, the share of individual 3's income that satisfies the condition imposed by proposition 1 is greater than the share that qualifies $\mathbf{a}_3$ as a Pareto-improving redistribution. Since $\mathbf{a}_3$ is a redistribution from individuals 1 and 2 to individual 3, a Pareto-improving redistribution from the noncontributors to the contributor of a standard public good is less likely to occur than the case observed in the absence of a weaker-link public good. We further note that if the share of individual 3's income is sufficiently small, a redistribution to individual 3 is Pareto improving.

The above statement can be explained as follows. Even in the presence of a weaker-link public good, a redistribution from the noncontributors to the contributor of the standard public good benefits the donors by increasing the provision of the standard public good. On the other hand, the cost involved with redistribution includes not only the income that is required for redistribution but also an indirect welfare cost. When such a redistribution reduces the income of a donor, he/she decreases the contribution to the weaker-link public good; this in turn harms another donor. Thus, the chances of a Pareto-improving redistribution are less likely in the presence than in the absence of a weaker-link public good. If the disparity in income is sufficiently small, the benefits arising from redistribution are large enough to exceed the direct income and indirect welfare costs associated with such redistribution, that is, the redistribution can be regarded as Pareto improving.

Let us consider the marginal change in income that improves the welfare of individual $i\,(i = 1, 2)$ without affecting those of other individuals. Substituting equations (6), (7), and (10) in equation (15), we have

$$\Delta v_{im} \frac{\partial M}{\partial u_i} = \frac{\alpha\,(\alpha + \beta + \gamma)}{[\alpha + (\beta/3)]\,[\alpha + (\beta/3) + \gamma]} \left[1 - \frac{(\beta/3)}{\alpha + \beta + \gamma} \left(\frac{M}{m_i} \right) \right] \leq 0, \quad i = 1, 2. \tag{19}$$

From equations (18) and (19), when we have

$$\frac{m_1}{M} > \frac{(\beta/3)}{\alpha + \beta + \gamma} \text{ and } \frac{m_3}{M} > \frac{(\beta/3) + \gamma}{\alpha + \beta + \gamma}, \tag{20}$$

no Pareto-improving redistribution can be accomplished by mere redistribution.[4] If condition (20) is satisfied, we have $\partial M/\partial u_i > 0$ for $i = 1, 2, 3$. Any Pareto-improving marginal

[4] As presented in Appendix C, this result can be directly proved using Tucker's theorem of the alterna-

change that improves the welfare of individual i by θ_i units is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$, i.e., $\sum_{i=1}^{3} \theta_i \mathbf{a}_i$,where $\theta_i \geq 0$ and $i = 1, 2, 3$, with a strict inequality for the welfare of at least one individual. Thus, any Pareto-improving marginal change necessarily requires an increase in the aggregate income and cannot be accomplished by any redistribution.

Comparing equation (20) with equation (11), it follows that if we have

$$\frac{m_1}{M} > \frac{(\beta/3)}{\alpha+\beta+\gamma} \text{ and } \frac{\gamma}{\alpha+(\beta/3)+\gamma} > \frac{m_3}{M} > \frac{(\beta/3)+\gamma}{\alpha+\beta+\gamma},$$

then the condition imposed by proposition 1 is satisfied; however, there exists no Pareto-improving redistribution.

In addition, we observe that if the share of individual i's income is sufficiently small, there is a Pareto-improving redistribution to individual i.

(FIGURE 2)

The relationship between initial income distribution and the possibility of a Pareto-improving redistribution is illustrated in figure 2. Every point in the triangle represents the initial distribution of the aggregate income, where the income of individual i is its distance from the side $O_j O_k$ $(i, j, k = 1, 2, 3;\ j \neq i, k;$ and $k \neq i, j)$. If an income distribution satisfies the condition $m_1 < m_2 < m_3$ and attains an equilibrium where individuals 1 and 2 contribute only to the weaker-link public goods, this distribution is represented by a point that belongs to one of the regions **a**, **b**, **c**, **d**, and **e**. In regions **d** and **e**, there is a Pareto-improving redistribution from individuals 2 and 3 to individual 1. In region **e**, there exists a Pareto-improving redistribution from individuals 1 and 3 to individual 2. In region **a**, there is a Pareto-improving redistribution from individuals 1 and 2 to individual 3. There is however no Pareto-improving redistribution in regions **b** and **c**; this contradicts the fact that the condition imposed by proposition 1 is satisfied in regions **a** and **b**.

Appendix A

In this appendix, we derive the Nash equilibrium of our model and also the condition that only one individual contributes to the standard public good. In our model, it is more straightforward to derive the Nash equilibrium by the replacement function approach than by the best-response function approach.[5] Hence, we adopt the former. Rearranging the terms in equation (3), we derive the replacement function for individual i; this relates i's optimal contribution to the standard public good with the total supply of the good, as

$$q_i^R(m_i, Q) = \begin{cases} m_i - \varepsilon Q & \text{if } m_i/\varepsilon > Q \\ 0 & \text{if } m_i/\varepsilon \leq Q \end{cases}, \tag{A1}$$

where $\varepsilon \equiv [\alpha + (\beta/3)]/\gamma$.

The equilibrium supply of the standard public good is determined by solving the following equation

$$Q = \sum_{i=1}^{3} q_i^R(m_i, Q) \tag{A2}$$

tives.

[5]For the replacement function approach, see Cornes and Hartley(2001,2003).

with respect to Q. Substituting equation (A1) in (A2) yields

$$Q = \begin{cases} m_1 + m_2 + m_3 - 3\varepsilon Q & \text{if } \quad Q < m_1/\varepsilon \\ m_2 + m_3 - 2\varepsilon Q & \text{if } m_1/\varepsilon \leq Q < m_2/\varepsilon \\ m_3 - \varepsilon Q & \text{if } m_2/\varepsilon \leq Q < m_3/\varepsilon \\ 0 & \text{if } m_3/\varepsilon \leq Q \end{cases}$$

Solving the above equation, we obtain the equilibrium levels of contributions to the standard public good as follows:[6]

1. When $m_1 > \varepsilon M/(1+3\varepsilon)$, we obtain

$$q_i = m_i - \frac{\varepsilon}{1+3\varepsilon} M;\ i = 1, 2, 3.$$

2. When $m_1 \leq \varepsilon M/(1+3\varepsilon)$ and $m_2 > \varepsilon m_3/(1+\varepsilon)$, we obtain

$$q_1 = 0, q_i = m_i - \frac{\varepsilon}{1+2\varepsilon}(m_2 + m_3);\ i = 2, 3.$$

3. When $m_2 \leq \varepsilon m_3/(1+\varepsilon)$, we obtain

$$q_1 = q_2 = 0,\ q_3 = \frac{1}{1+\varepsilon} m_3.$$

Hence, there exists a Nash equilibrium at which individuals 1 and 2 contribute only to the weaker-link public good if and only if we obtain the following result:

$$m_2 \leq \frac{\varepsilon}{1+\varepsilon} m_3 = \frac{\alpha + (\beta/3)}{\alpha + (\beta/3) + \gamma} m_3.$$

Appendix B

Vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are derived by the following equations:

$$\begin{aligned}
\mathbf{a}_1 &\equiv \frac{1}{\Delta v_{1m}} \begin{pmatrix} 1 - (v_{23}x_3' + v_{2Q}Q')\, v_{32}x_2' \\ -v_{21}x_1' + (v_{23}x_3' + v_{2Q}Q')\, v_{31}x_1' \\ -v_{31}x_1' + v_{32}x_2'v_{21}x_1' \end{pmatrix}, \\
\mathbf{a}_2 &\equiv \frac{1}{\Delta v_{2m}} \begin{pmatrix} -v_{12}x_2' + (v_{13}x_3' + v_{1Q}Q')\, v_{32}x_2' \\ 1 - (v_{13}x_3' + v_{1Q}Q')\, v_{31}x_1' \\ -v_{32}x_2' + v_{31}x_1'v_{12}x_2' \end{pmatrix}, \\
\mathbf{a}_3 &\equiv \frac{1}{\Delta v_{3m}} \begin{pmatrix} -(v_{13}x_3' + v_{1Q}Q') + v_{12}x_2'(v_{23}x_3' + v_{2Q}Q') \\ -(v_{23}x_3' + v_{2Q}Q') + v_{21}x_1'(v_{13}x_3' + v_{1Q}Q') \\ 1 - v_{21}x_1'v_{12}x_2' \end{pmatrix}.
\end{aligned}$$

Substituting equations (6), (7), and (10) in the above equations, we obtain the following

[6] When $\gamma m_3/[\alpha + (\beta/3)] \leq Q$, we obtain $m_3 \leq 0$; this contradicts the assumption that $m_3 > 0$.

values of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, respectively

$$\mathbf{a}_1 = \frac{\alpha}{\Delta v_{1m}\left[\alpha+(\beta/3)\right]\left[\alpha+(\beta/3)+\gamma\right]}\begin{pmatrix} \alpha+(2\beta/3)+\gamma \\ -(\beta/3)(m_2/m_1) \\ -(\beta/3)(m_3/m_1) \end{pmatrix},$$

$$\mathbf{a}_2 = \frac{\alpha}{\Delta v_{2m}\left[\alpha+(\beta/3)\right]\left[\alpha+(\beta/3)+\gamma\right]}\begin{pmatrix} -(\beta/3)(m_1/m_2) \\ \alpha+(2\beta/3)+\gamma \\ -(\beta/3)(m_3/m_2) \end{pmatrix},$$

$$\mathbf{a}_3 = \frac{\alpha}{\Delta v_{3m}\left[\alpha+(\beta/3)\right]^2}\begin{pmatrix} -\left[(\beta/3)+\gamma\right](m_1/m_3) \\ -\left[(\beta/3)+\gamma\right](m_2/m_3) \\ \alpha+(2\beta/3) \end{pmatrix}.$$

It is straightforward to conclude that the ith component of vector $\mathbf{a}_i$ is positive, while the other components of the vector are negative.

Appendix C

From equation (13), we can define a Pareto-improving redistribution as a vector, **dm**, such that

$$\mathbf{du} = \mathbf{Adm} \geq \mathbf{0}, \mathbf{Adm} \neq 0, \mathbf{b}^t\mathbf{dm} \leq \mathbf{0}.$$

Further, using Tucker's theorem of the alternatives, we observe that exactly one of the following hold

1. There exists a Pareto-improving redistribution, i.e., **dm**.
2. There exist a vector $\mathbf{d}_1 \in \mathbf{R}^3$ and a scalar d_2 satisfying

$$\mathbf{A}^t\mathbf{d}_1 - d_2\mathbf{b} = 0, \mathbf{d}_1 > 0, d_2 \geq 0. \quad \text{(C1)}$$

Equation (C1) can be rewritten as $\mathbf{d}_1^t\mathbf{A} = d_2\mathbf{b}^t$. Solving this equation with respect to $\mathbf{d}_1$, we obtain

$$\mathbf{d}_1^t = d_2\mathbf{b}^t\mathbf{A}^{-1} = d_2\left(\mathbf{b}^t\mathbf{a}_1, \mathbf{b}^t\mathbf{a}_2, \mathbf{b}^t\mathbf{a}_3\right).$$

Hence, when we have $\mathbf{a}_i^t\mathbf{b} > \mathbf{0}$ for any $i = 1, 2, 3$, the second case holds true. Thus, we conclude that there is no Pareto-improving redistribution.

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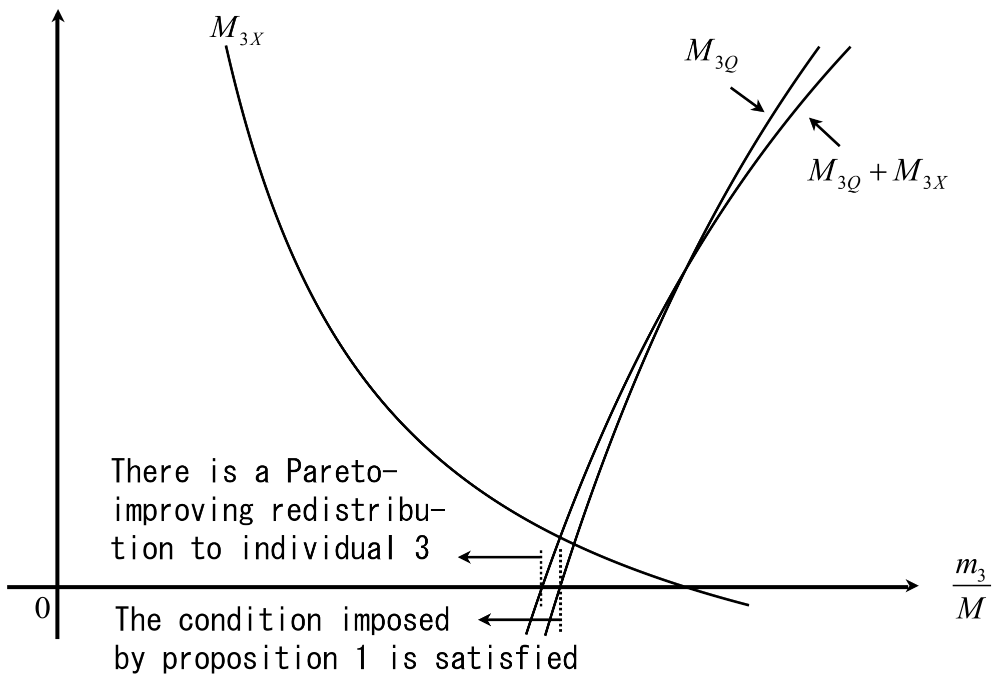

Figure 1: Pareto-improving marginal change in income and the share of individual 3's income

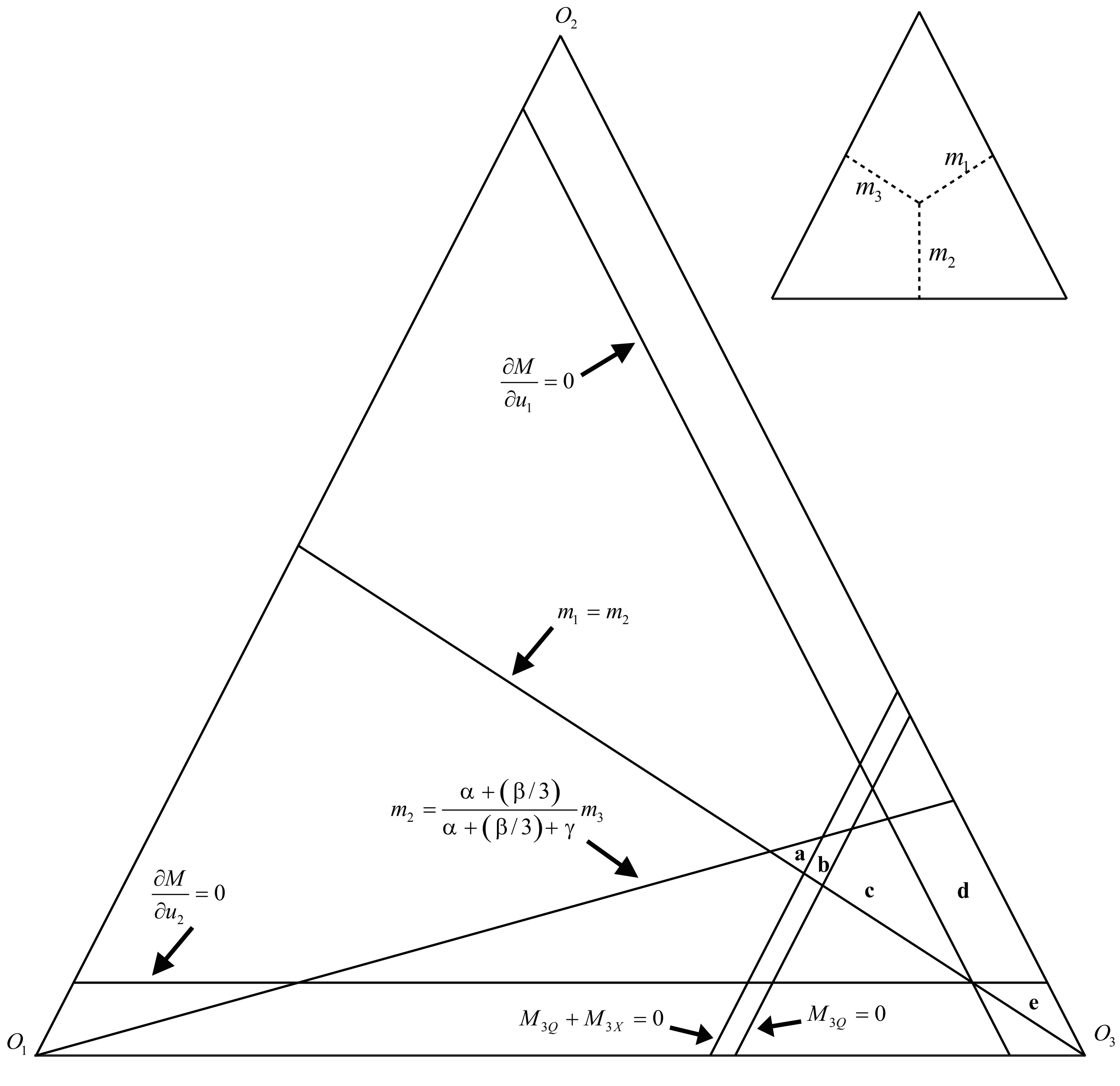

Figure 2: Initial income distribution and the possibility of Pareto-improving redistribution