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Examples of Attractor Switching in Simple Co-Adaptive Economic Production Networks

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Abstract

In this paper we describe a simple model of a dynamical evolving production network. The model is based on a combination of von-Neumann's neoclassical model of production and on autocatalytic chemical reaction networks. The rate of economic production is described by a Leontief type minimum production function. In this paper we concentrate on describing the model in some detail and illustrate its behaviour with reference to a simple system - the single production process in a fixed external environment. We show that even such a simple system can be expected to show complex non-equilibrium dynamics, with endogenous cycles, price bubbles and crashes, similar to real business cycles. This dynamical complexity is shown to originate from the fact that the process feeds back onto the prices of its supply products in a highly non-linear way producing threshold oscillations caused by a novel type of attractor switching produced by the Leontief type minimum condition.

1 Introduction.

In this paper we describe a non-linear dynamical model of an evolving production network which we hope describes in a very simple way the dynamics and evolution of a production economy. Our motivation is to understand the dynamical origin of non-equilibrium macroeconomic phenomena such as business and economic cycles, economic growth and recession, and price fluctuations. In this paper we concentrate on describing and explaining this new model and the novel dynamical phenomenon it shows, especially the origin of complex

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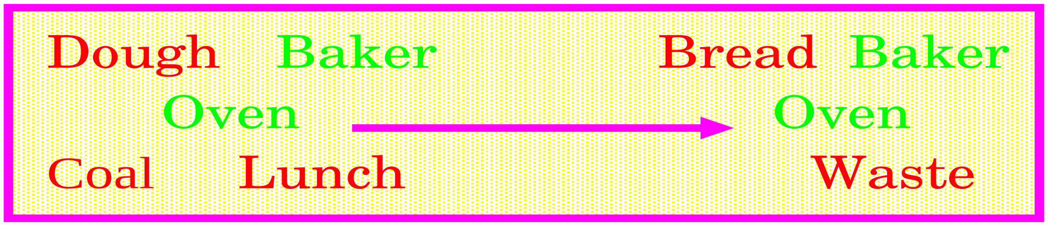

Fig. 1. An economic production process. The catalysts baker and oven are shown green and the other products red.

cyclic dynamics, through a novel type of attractor switching, and on their economic significance.

The model is based on von-Neumann's neoclassical model of economic production [Von-Neumann (1945), Morishima (1970)], which we now briefly review, and on catalytic reaction networks [Kaneko et al.(1997), Furusawa et al.(2001)].

In von-Neumann's model of economic production each good is produced jointly with certain others like a chemical reaction. A process of production converts one bundle of goods, including capital equipment, into another bundle of goods, including the capital equipment. Capital goods are therefore treated something like catalysts in a chemical reaction, reformed at the end of the reaction in conserved quantities. Therefore the capital equipment employed is included in both the bundle of inputs as well as the bundle of outputs and each process can therefore be described by fixed input stochiometric ratios and fixed output stochiometric ratios just like a chemical reaction. Capital goods at different stages of 'wear and tear' can be treated as different goods. Furthermore it is assumed that consumption of goods takes place only through the processes of production which include all necessities of life consumed by workers and all income above necessities of life is reinvested in production. In an economy there are a fixed amount of processes and products representing all the technologically possible transformations, although not all processes need be active. Each process can be considered to be of unit time duration and longer processes can be broken down into a number of intermediate processes, with intermediate products if necessary.

An example of an economic production process is illustrated in Fig.1. In this process, which converts dough into bread, there are two catalysts, the baker and the oven and two consumed 'energy' products. Of course like a chemical reaction the process can be broken down into steps. For example the baker eats her lunch and becomes an 'activated' baker. Then she puts the coal and dough in the oven and subsequently decays to deactivated baker and waste. The activated oven consumes the coal and deactivates forming bread and waste.

When many processes are coupled together a production economy becomes analogous to a catalytic reaction network, an example of which is shown in Fig.2(a). Intermediate products such as 'dough-in-oven' and 'heat' are shown. Here we also include an environment where some products may be continuously supplied and others continuously demanded. The environment may be another country or the natural environment; trained workers may emmigrate for example, and sunlight and metal-ore may be supplied for free. Note we can also include pure consumption processes in this framework, as simply reactions without catalysts. Autocatalytic processes such as (untrained worker + bread → two × untrained worker), will also exist.

The original von-Neumann model however is a static equilibrium model and analysis proceeds by considering relations which must hold at equilibrium. For example suppose the system has M processes labeled $i = 1, ..., M$ and N products labeled $i = 1, ..., N$. Suppose the input stochiometric ratios are given by a_{ij} for process i input of product j and b_{ij} for process i output of product j, and $z(t)$ is an N dimensional vector representing the 'intensity' of processes i and $P(t)$ is a M dimensional vector describing prices of products j. Also suppose the present interest rate is given by $\beta(t)$. Then in equilibrium there can be no process which yields a return greater than the present interest rate for under perfect competition positive profits would attract competitors to use the same process so that prices of factors would rise. Therefore von-Neumann obtained,

$$BP(t+1) \leq \beta(t)AP(t) \tag{1}$$

where $B = (b_{ij})$ and $A = (a_{ij})$.

Next if a process yields negative profits after payment of interest it will not be used, and its intensity is zero, therefore,

$$z(t)BP(t+1) = \beta(t)z(t)AP(t). \tag{2}$$

Since each process is of unit time duration, the components of the vector $z(t-1)B$ give the amounts produced at time t, while those of the vector $z(t)A$ give the amounts of input used up in production at time t. It is impossible to consume more of a good in the production processes than is available, so von-Neumann gets,

$$z(t-1)B \geq z(t)A. \tag{3}$$

Finally, in equilibrium those goods that are overproduced will be free goods and zero prices are charged for them. This implies,

$$z(t-1)BP(t) = z(t)AP(t). \tag{4}$$

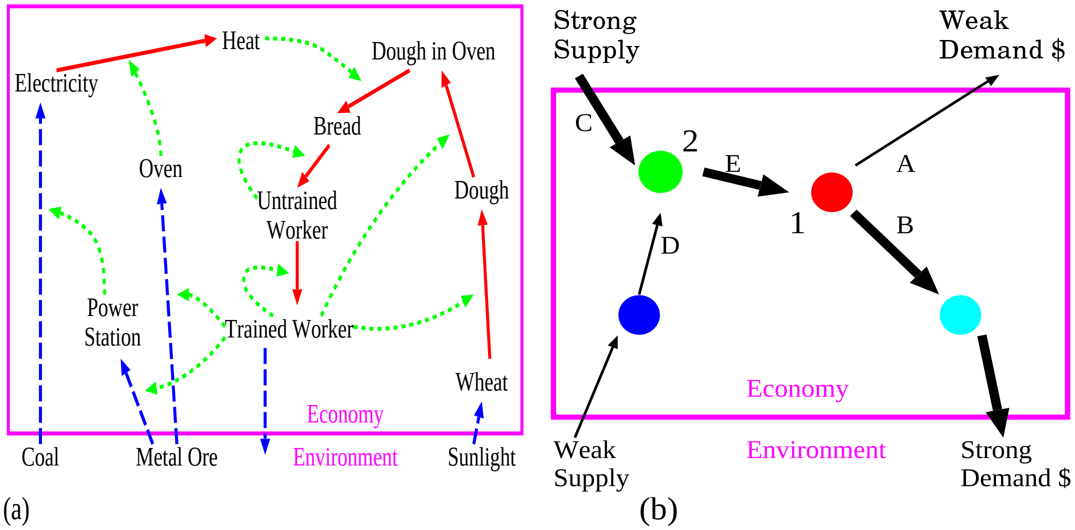

Fig. 2. (a) An VNM Economy describing a fixed set of production processes and products. Red lines are production flows and green dotted lines are catalytic effects. Some processes are autocatalytic as shown. The system may be closed or open. In an open system some products are supplied from the environment and others are demanded, shown as blue dashed lines. (b) An adaptive network. Coloured circles denote products transformed by production processes which are shown as arrows. In this simple example each process is assumed to only have one input product and one output product and catalysts are not shown for convenience. Product moves from external input supply to external output demand in the positive arrow direction. Funds (i.e. money) move from the external output demand to the external input supply in the negative arrow direction. Arrow size indicates the strength of the process, i.e. its value, that is availble funds, and therefore its share of the available input material. Processes A and B both demand the red product 1 and are therefore in competition for it. Process B has access to larger funds than process A so it takes the larger share. Similar processes C and D both supply the green product 2 and so are in competion for funds coming from process E. Process C makes green product 2 from an external product in larger supply than process D does. Process C then takes a larger share of the available funds and has a larger size. In this way some pathways are *strengthed* while occurs may go *bankrupt* completely. The path from strong supply to strong demand forms a production chain, i.e an assembly line. In this way the network *adapts* or internally evolves to fulfil the *function* of changing its external supplies into external demands in the most efficient way possible.

Von-Neumann defines equilibrium to be a state of balanced growth where prices and rate of interest are constant over time and intensities of production grow or decay at constant geometric rates.

However it is clear that a real economy is always far from equilibrium, even without considering technological evolution. While the above conditions Eq.1-4 may be relevant for equilibrium economic analysis, in practice they may not hold for a real far from equilibrium economy. In such a non-equilibrium state for example real production processes do make profit and competitor processes cannot appear immediately which means condition Eq.1 doesn't hold in reality. Concerning Eq.2, while it is clear than continuously unprofitable processes will eventually go bankrupt and their intensities will go to zero, this will take time and there will also be times when generally profitable processes have unprofitable periods but still have non-zero intensity. Eq.3 however, should be obeyed also far from equilibrium but Eq.4 is highly debatable. Indeed we will show that in our model some of these conditions dynamicaly appear in a natural *emergent* way.

Furthermore in a real economy, production processes, that is companies, and economic agents in general, have a *value*, which allows them access to funds i.e. money. One reason that economies also use money is simply that processes must have some means to store value, in the form of money, between selling of produced output products and the buying of new input products. For example if production process A requires goods produced by production process B, but production process B does not require goods from A, so that direct swapping is not possible, some storage mechanism is required. The simplest form of such storage is money. (In this sense money is somewhat similar to ATP in biological systems, as an energy store.) In reality a production process will not hold its own money but will go to a bank and get a loan to buy the goods necessary for its further production. The bank will give the process money in proportion to the current valuation of the process' holding of real goods. When these goods are sold (to other processes paying money borrowed from their banks) the production process will then return the money to the bank. Money allows a process to form a demand for a product, dependent on its current valuation, and allows each product to have its own real value which can be directly compared to the value of other products. It is almost impossible to imagine a real functional market economy without money, in the same way it is impossible to imagine biological processes without an energy store.

More importantly for our model, money exists as the *medium* through which processes evolve or adapt. It is through money, i.e. access to funds from a bank, that the economy can become an adaptive network as described in Fig.2(b), and we will go into this in more detail when we describe the model.

One source of disequilibrium in a real economy is the following allocation problem. Consider an economy with M production processes each having current value $F_i(t)$, i.e. each having access to funds $F_i(t)$, and N products each with a total stock of $S_j(t)$ in the whole economy. The problem is to allocate product prices $P_j(t)$ and production intensities $Z_i(t)$ so that all processes are satisfied. The relevant equations are,

$$S_j(t) = \sum_{i=1}^{M} Z_i(t) a_{ij}, \tag{5}$$

$$F_i(t) = \sum_{j=1}^{N} Z_i(t) a_{ij} P_j(t). \tag{6}$$

In general finding $P_j(t)$ and $Z_i(t)$ to satisfy this system for arbitrary given $F_i(t)$ and $S_j(t)$ will not be possible. For example in the case that $N > M$ Eq.5 may not even have a solution. Even if a solution exists it may not be in the real positive quadrant for the $Z_i(t)$, as required by reality. If processes only allocate a proportion of their available funds then we will not need to fulfill Eq.6, but this will not be 100% efficient, since we expect a competitive process will wish to invest all it can in production, as is also assumed in the VNM. Furthermore even if a perfect utility maximizing solution is possible in principle, there is no reason to think the system will necessarily continuously dynamically attain such a state as $F_i(t)$ and $S_j(t)$ vary non-linearly. In fact we are led to the conclusion that in general production processes will not obtain input products in exactly the stochiometric ratios they require for production to proceed without any leftover input, or alternatively they will have unsold output products, or alternatively they will have not been able to utilize all their available funds. This means an economy will usually not be in equilibrium and there will be processes with too much of some inputs and too little of others. This may be an origin of unemployment, where machinery or labour may be idle for lack of input material.

In our new model which we describe in this paper, we try to take a natural and realistic 'bottom-up' approach to describing a dynamical production economy. We try to make the simplest model possible which shows an evolving structure where production processes with inputs in strong supply and outputs in strong demand become fitter and gain in intensity, while the intensity of other processes may dynamically decay to zero, i.e. they go 'bankrupt', if no processes require their output products. In this way a production economy will be a dynamically evolving object represented as an *adaptive network of production processes,* similar in structure to a neural network, as described in Fig.2(b). Similarly we expect products required by many fit processes and supplied by few to have emergent high prices while other products with large supply and less demand will be cheaper.

Through such a bottom-up physics-style approach we hope to understand the origin and general features of economic growth, unemployment and price instability. Indeed this approach has also been influenced by recent work aimed at understanding cell biology, cell growth and diversification based on general characteristics of the dynamics of autocatalytic chemical reaction networks [Kaneko et al.(1997)]. In these works it is shown that pure dynamical effects, such as the presence of multiple attractors, can strongly influence the course of development of a cell including its final specialized state as well as whether it survives or ultimately dies. Indeed we hope in the end to be able to describe a production economy in an analogous (but somewhat more complex) way to this. We expect that in economics too, dynamical effects, such as path dependence, will play a large role in the development and specialization of a country's economy, as well as in its final state of success or failure and it is to explore such important effects that we have initiated this model and line of research.

2 Model

In this model there are a fixed amount of feasible processes M labeled $i = 1, ..., M$ and a fixed amount of possible products N labeled $j = 1, ..., N$. Of course some of these processes may be bankrupt due to unprofitability and some products may not be being produced at any given time. This network simply represents all the *technologically feasible* processes and is considered fixed so we are not concerned with technological evolution and innovation in this simplest first model. Each process i has fixed input stochiometric ratios a_{ij} and fixed output stochiometric ratios b_{ij} of products j representing the quantities required and produced by the process. Each process is also considered to have dynamic *supplies* of products $S_{ij}(t)$. Furthermore each process also has access to dynamic funds $F_i(t)$, measured in currency, representing the process' value or its *size.*

Our model [Ponzi et al(2003)], is defined in continuous time by the following system,

$$\frac{dS_{ij}(t)}{dt} = -S_{ij}(t) + \frac{F_i(t)\sigma_{ij}(t)}{p_j(t)} + \alpha(b_{ij} - a_{ij})Min_k(\frac{S_{ik}(t)}{a_{ik}}) \tag{7}$$

$$\frac{dF_i(t)}{dt} = -F_i(t) + \sum_j S_{ij}(t)p_j(t) \tag{8}$$

where

$$p_j(t) = \frac{\sum_k F_k(t)\sigma_{kj}(t) + D_j^{ext}}{\sum_k S_{kj}(t) + S_j^{ext}} \tag{9}$$

$$\sigma_{ij}(t) = \gamma\frac{p_j(t-\tau)a_{ij}}{\sum_j p_j(t-\tau)a_{ij}} + (1-\gamma)a_{ij} \tag{10}$$

While this system looks rather formidable, in fact it is quite simple. Eq.8 describes the change in process i value, i.e. its funds $F_i(t)$, with time t. Eq.7 describes the change in process i supplies of product j, $S_{ij}(t)$. In these equations $p_j(t)$ is the product j value, or price, given in continuous time by Eq.9.

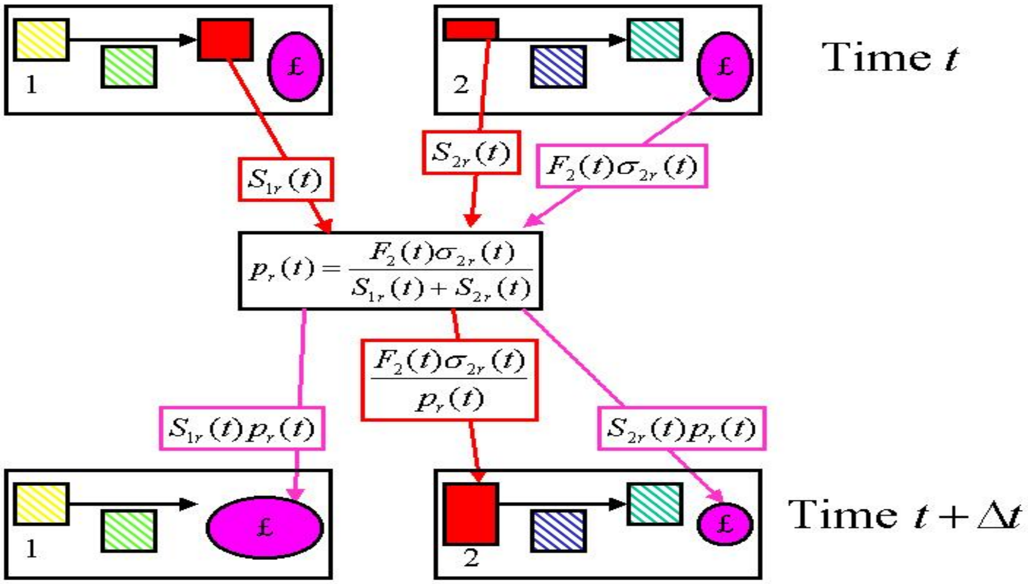

Fig. 3. The Marketing phase; product and process revaluation. This schematic shows two processes denoted 1 and 2 and the price setting market for the product 'red' denoted r. Each process has one input product, one catalyst and one output product. Process 1 produces product r but does not consume it. Process 2 consumes product r. In continuous time both processes continuously submit their supplies of product r, $S_{1r}(t)$ and $S_{2r}(t)$ to market. Since process 2 requires product r it also submits its *demand* for product r, $F_2(t)\sigma_{2r}(t)$ in \$ where $\sigma_{2r}(t) < 1$ represents the proportion of its total value $F_2(t)$ process 2 has allocated to buying product r. The product value $p_r(t)$ is then simply formed as total demand divided by total supply, so that $p_r(t) = F_2(t)\sigma_{2r}(t)/(S_{1r}(t) + S_{2r}(t))$. The submitted funds $F_2(t)\sigma_{2r}(t)$ are then reshared between the two processes according to their supplies multiplied by the price so that process 1 receives $S_{1r}(t)p_r(t)$, while process 2 receives $S_{2r}(t)p_r(t)$. Notice that process 2 receives some of its own submitted funds $F_2(t)\sigma_{2r}(t)$ *back*, i.e. $S_{2r}(t)p_r(t)$. This is because it has left over input supply $S_{2r}(t)$ of product r which has not yet been processed, but nevertheless, must be considered in the price formation and resharing process. Since process 1 does not submit any demand, process 2 receives all the supply of product r as its demand divided by the price $F_2(t)\sigma_{2r}(t)/p_r(t)$. In this way products are continuously revalued and continuously reshared while processes are similarly revalued according to their changing value $F_i(t)$.

The model has two dynamical parts (i) marketing (ii) processing, which are both considered to be happening simultaneously and continuously.

(i) Marketing. The marketing part describes continuous product and process valuation and is illustrated in Fig.3. The first two terms on the RHS of Eqns.8,7 describe this part. In the first term on the RHS of Eq.7 process i submits all its current holdings of product j, $S_{ij}(t)$ to market. Simultaneously process i forms *demands*, measured in money (\$), for the products it needs for processing, by allocating all its current funds $F_i(t)$ between the different input products it needs to buy. That is process i allocates $F_i(t)\sigma_{ij}(t)$ \$ to buy product j where $\sigma_{ij}(t) < 1$ is a demand coefficient. We will say more about the demand coefficients below but the most important point is that the demand coefficients are such that $\sum_j \sigma_{ij}(t) = 1 \; \forall i$. Therefore processes allocate exactly all of their funds $F_i(t)$ to buying input products and submit them to market and this is shown in the first term on the RHS of Eq.8. The correct market price is then formed as (Total Demand for product j) divided by (Total Supply of product j) as in Eq.9, where D_j^{ext} is a possible external demand for product j in \$ and S_j^{ext} is a possible external supply of product j. Products and funds are then reshared among the processes according to the second terms on the RHS of Eqns.8,7. That is processes receive funds as $S_{ij}(t)p_j(t)$ while they receive products as $F_i(t)\sigma_{ij}(t)/p_j(t)$.

There are a few points here: (i) In the determination of the real correct market price for product j, *all* supplies $S_j(t) = \sum_i S_{ij}(t)$ of product j in the economy at time t, and *all* funds $\sum_i F_i(t)\sigma_{ij}(t)$ available to buy product j are considered. In this model, based on ideas from competive evolution, processes would not be as competitively fit as possible if they either held back supplies from market or they did not attempt to use all their possible available funds. (ii) Processes receive funds, in the simplest way, in linear proportion to their supplies $S_j(t)$, and processes receive supplies, in the simplest way, in linear proportion to their funds $F_i(t)$. (iii) This marketing is considered to occur continuously and therefore amounts to a continuous product revaluation and a continuous process revaluation via the changing process value $F_i(t)$.

Since processes use all of their funds in buying input products, this means that processes will not in general receive products at rates in the correct stochiometric ratios necessary for perfect production. Indeed due to the *continuous* time market revaluation, unused excess input can simply be considered as unused funds since this unused excess input is sent straight back to market for revaluation and converted back into funds. In fact funds and the supplies they represent are interchangeable at the current market price. As explained above, it is necessary to include funds in the model, however, as it is in a real economy, to facilitate the formation of simultaneous market prices for several different products when a process needs to buy several different inputs for processing simultaneously, and to facilitate competition between processes when several processes try to buy the same product simultaneously. Funds therefore become the medium for the process of competitive evolution. We believe this marketing mechanism is the simplest possible which retains this necessary structure.

To labour the point somewhat, the funds should be thought of as the value of the process that it can utilise for buying products, rather than as an *extra* to its supplies. This is seen by considering the equilvalent time discretized system. We get,

$$F_i(t) = \sum_j S_{ij}(t-1)p_j(t-1) \tag{11}$$

from Eq.8. Therefore the $F_i(t)$ which appears in Eqs.7,9 is simply a continuous time shorthand version of the process value in the previous timestep Eq.11, and in discrete time the system could have been rewritten by substituting this equation to become a time-delay system. But this description would be much less intuitive.

(ii) Processing. For this we have to consider how a process' processing rate depends on its supplies of input products. In chemistry this is given by the Law of Mass Action. This states that the reaction rate increases when the concentration of any of the reagents increases. This law which depends on the consideration of random collisions as well as on the notion of volume is obviously inapplicable in an economic reaction context. In fact, as described by Leontief [Leontief (1941)], and pointed out earlier by Cassel [Cassel (1918)], the rate of an economic reaction is given by the quantity of the *minimum* supply possessed by the process at any time t.

For example consider the reaction depicted in Fig.1. Suppose we have one baker, plenty of ovens and a high rate of supply of dough. Changing the rate of supply of dough, will not affect the production rate since the baker is already working at full pace. The dough will simply pile up. However hiring more bakers, while still having plenty of ovens will of course increase the rate of production. Similarly suppose we have plenty of bakers and a low dough supply rate. In any given time period the same quantity of bread will be produced even if the process hires more bakers or fires a few. While there are excess bakers some of them will simply sit around part of the time or they will all work slowly. Therefore we expect that the processing rate $R_i(t)$ will be given by the following Leontief type expression,

$$R_i(t) = Min_k(\frac{S_{ik}(t)}{a_{ik}}) \tag{12}$$

where $Min_k(x_k)$ means the minimum over x_k. This appears as the third term on the RHS of Eq.7. The parameter α is a relative timescale of processing and market resharing.

In fact, we can generalise this minimum condition Eq.12 somewhat to the following function,

$$R_i(t) = M(x_k) = (\sum_k x_k^{-r})^{-1/r} \tag{13}$$

which becomes the minimum Eq.12 in the limit $r \rightarrow \inf$. In the following we only consider the case where r is large enough for Eq.12 and Eq.13 to be considered equivalent.

The only part of the model left to explain is the way a process allocates its accessable funds into demands for its various inputs, i.e. how to determine $\sigma_{ij}(t)$. There are many ways to determine these. One could consider strategic considerations such as using game theory. One may consider them simply random numbers, normalized, as explained above. However, the simplest possible presciption is that the process divides its value according to its input stochiometric ratios a_{ij}. All things considered, if the process has no information concerning the price $p_j(t)$ to be formed in the market, this would seem to be the most efficient prescription, where in the long run the process would obtain input supplies at the rates necessary for perfect production without excess unused input, or equivalently excess unused value. If input products have consistently different prices the process may wish to take a previous price into account when allocating its value. In this case $p_j(t-\tau)a_{ij}/\sum_j p_j(t-\tau)a_{ij}$ is the appropriate division of value since the numerator is the per unit of processing cost of input product j for process i. τ would be some previous time, such as yesterday, or last month, or more likely the most recent known price. Alternatively a process may wish to take into a account an average over previous prices.

In this paper we consider the prescription Eq.10 where γ is a parameter which quantifies feedback from a previous price. When $\gamma \neq 0$ we consider the equilvalent time discretized system (as will be described) and set $\tau = 1$. I.e. processes look at the last known price $p(t-1)$ to allocate their current funds into demands. If $\gamma = 1$ this means that agents consider $p(t-1)$ to be the best indicator of the next price $p(t)$ to be formed in the market by their collective action, and allocate their funds accordingly. Since this observation of the previous price and it's use as an estimator of the next unknown price $p(t)$ is a kind of rationality, we refer to the system with $\gamma \sim 1$ as "rational", but the word should only be understood in this simplest very restricted sense, i.e. the ability to "know" a price.

3 Results

We have gone to some length in the above to motivate the description of an economy as an adaptive network of economic processes. Indeed in general we expect complex macroeconomic behaviour to emerge when many simple microeconomic processes are coupled so that they can adapt to each other. However to understand such a system we first have to understand the dynamics of a single process in a *fixed* environment, without other *coadaptive* processes.

Furthermore while such a fixed environment system may seem unrealistic it can even itself be considered a *first approximation* to a complete production economy, as we will describe.

With this in mind we first consider a simple single process example and then we consider a chain of such production processes, illustrating the appearance of a multiple timescale limit cycle causing complex unpredictable production dynamics.

3.1 Single Process in Fixed Environment

To illustrate the meaning of the parts of this model and its dynamical behaviour we first consider the simplest (non-trivial) example possible. This is a one input product I, one catalyst product C, one output product O system in a fixed external environment. Indeed, as noted, such a system may represent a highly idealized single country, where the input product is imported at a fixed rate from the external environment, the output product is demanded by other countries at a fixed rate and where the catalyst, which changes the import product into the output product, may simply represent the labour force, which may also freely flow to and from the external environment. We consider the production process perfectly efficient, and neglect the labour consumption of life necessities and waste production, as well as 'wear and tear' of the labour force.

Clearly this is a gross idealization. Nevertheless we believe it is still a relevant system to study and indeed captures the minimal essential characteristics of such a situation. Indeed we show here that this simple system shows surprisingly complex dynamics, especially given the fact the environment is fixed. In particular we show that the characteristic dynamical behaviour is a limit cycle oscillation, which may even be identified with an endogenous business cycle. We show that this limit cycle is a type of threshold oscillation produced by a feedback of the process onto the prices of its supply products and that this in turn causes endogenous sudden price bubbles and crashes. Furthermore we show that the production process shows intuitively correct dynamical behaviour. In particular we show (a) that the process always tries to keep its catalyst and input supplies balanced in the correct stochiometric proportions, (b) that when a process has excess of some input product, the price of that product goes to zero, while the price of rate controlling minimum product bubbles, (c) that the production process will be unstable and dynamically 'go bankrupt' under certain conditions defined by the external environment and (d) that if the production process is feasible it acts on its external supplies and demands so as to equalize equilbrium (average) input and output prices.

The system is described by the input supply $S_I(t)$, the catalyst supply $S_C(t)$ and the output supply $S_O(t)$ as well as the process value $F(t)$. Since we are considering the process perfectly efficient the stochiometric ratios are $a_I = 1 - a_C$ and $b_O = 1 - b_C$ where also $a_C = b_C$, and $b_I = a_O = 0$. There are also the environment variables S_I^{ext}, S_C^{ext}, and S_O^{ext} describing fixed external supply rates and D_I^{ext}, D_C^{ext}, and D_O^{ext} describing fixed external demand rates.

3.1.1 Oscillation without price feedback

In order to understand the dynamics as simply as possible we first consider the simplified system obtained by setting the price feedback parameter $\gamma = 0$. Indeed this system shows much the same qualitative dynamical behaviour as the full system with $\gamma > 0$, but is much easier to analyse. Later we remove this restriction and consider the more economically relevant case where $\gamma \sim 1$.

The process is described by,

$$\begin{aligned}
\frac{dS_I(t)}{dt} &= -S_I(t) + F(t)a_I\frac{S_I(t) + S_I^{ext}}{F(t)a_I + D_I^{ext}} - \alpha a_I Min(\frac{S_C}{a_C}, \frac{S_I}{a_I}) \\
\frac{dS_C(t)}{dt} &= -S_C(t) + F(t)a_C\frac{S_C(t) + S_C^{ext}}{F(t)a_C + D_C^{ext}} \\
\frac{dS_O(t)}{dt} &= -S_O(t) + \alpha b_O Min(\frac{S_C}{a_C}, \frac{S_I}{a_I}) \\
\frac{dF(t)}{dt} &= -F(t) + \frac{S_I(t)(F(t)a_I + D_I^{ext})}{S_I(t) + S_I^{ext}} + \frac{S_C(t)(F(t)a_C + D_C^{ext})}{S_C(t) + S_C^{ext}} + \frac{S_O(t)D_O^{ext}}{S_O(t) + S_O^{ext}}
\end{aligned} \tag{14}$$

In order to understand the meaning of this system we first consider two further simplifications. (i) Zero external demand rates, (ii) zero external supply rates.

(i) We suppose, for example, at first, that there is no external demand for input product or catalyst, i.e. $D_I^{ext} = D_C^{ext} = 0$. This means the process has no competition when buying its input and catalyst products. Then the process value $F_i(t)$ becomes irrelevant and the process receives all the available supply anyway. In the case $\frac{S_C(t)}{a_C} < \frac{S_I(t)}{a_I}$ so that the catalyst supply $S_C(t)$ is the minimum, and governs the processing rate, Eqs.14 become simply,

$$\begin{aligned}
\frac{dS_I(t)}{dt} &= S_I^{ext} - \alpha\frac{a_I}{a_C}S_C(t) \\
\frac{dS_C(t)}{dt} &= S_C^{ext} \\
\frac{dS_O(t)}{dt} &= -S_O(t) + \alpha\frac{b_O}{a_C}S_C(t)
\end{aligned}$$

Now it is clear that S_I^{ext} and S_C^{ext} are to be thought of as *rates*. The input supply $S_I(t)$ increases at a rate S_I^{ext} and is consumed at a rate $\alpha\frac{a_I}{a_C}S_C(t)$, (in the case here, where $\frac{S_C(t)}{a_C} < \frac{S_I(t)}{a_I}$), i.e. at a rate proportional to the quantity of catalyst $S_C(t)$. While the catalyst supply simply grows at a rate S_C^{ext}. These equations are easily solved exactly so that,

$$\begin{aligned}
S_I(t) &= S_I(0) + S_I^{ext}t - \alpha\frac{a_I}{a_C}(S_C(0)t + \frac{1}{2}S_C^{ext}t^2)\\
S_C(t) &= S_C(0) + S_C^{ext}t\\
S_O(t) &\sim \alpha\frac{b_O}{a_C}S_C^{ext}t
\end{aligned}$$

Therefore the catalyst supply $S_C(t)$ grows linearly and the input supply $S_I(t)$ has a quadratic behaviour with a peak. The output supply $S_O(t)$ simply follows the catalyst supply level since it is the minimum. Of course since $S_I(t)$ will eventually decrease, eventually these equations will cease to hold since the Leontief type minimum processing condition will *switch* when $\frac{S_C(t)}{a_C} > \frac{S_I(t)}{a_I}$. In this case the appropriate equations become,

$$\begin{aligned}
\frac{dS_I(t)}{dt} &= S_I^{ext} - \alpha S_I(t)\\
\frac{dS_C(t)}{dt} &= S_C^{ext}\\
\frac{dS_O(t)}{dt} &= -S_O(t) + \alpha\frac{b_O}{a_I}S_I(t)
\end{aligned}$$

so that $S_C(t)$ continues to grow linearly but now $S_I(t) \sim S_I^{ext}$ and $S_O(t) \sim \alpha\frac{b_O}{a_I}S_I^{ext}$. In this case where there is no external catalyst demand, $D_C^{ext} = 0$, this is now the final state. Indeed since there is no competition for the catalyst, $S_C(t)$ continues to grow, but $S_I(t)$ is fixed, so there is no more minimum switching.

(ii) On the other hand now suppose that rather than all the external demands in Eqs.14 being zero, consider the case that all the external supplies are zero, i.e. $S_I^{ext} = S_C^{ext} = S_O^{ext} = 0$, so that the process is the sole owner of these 3 products. (This is not possible in practice however since the process would soon consume all its input product $S_I(t)$ without external supply S_I^{ext}.) Then, since the input ratios are normalized ($a_I + a_C = 1$) the value equation in Eq.14 simply becomes,

$$\frac{dF(t)}{dt} = D_I^{ext} + D_C^{ext} + D_O^{ext} \tag{15}$$

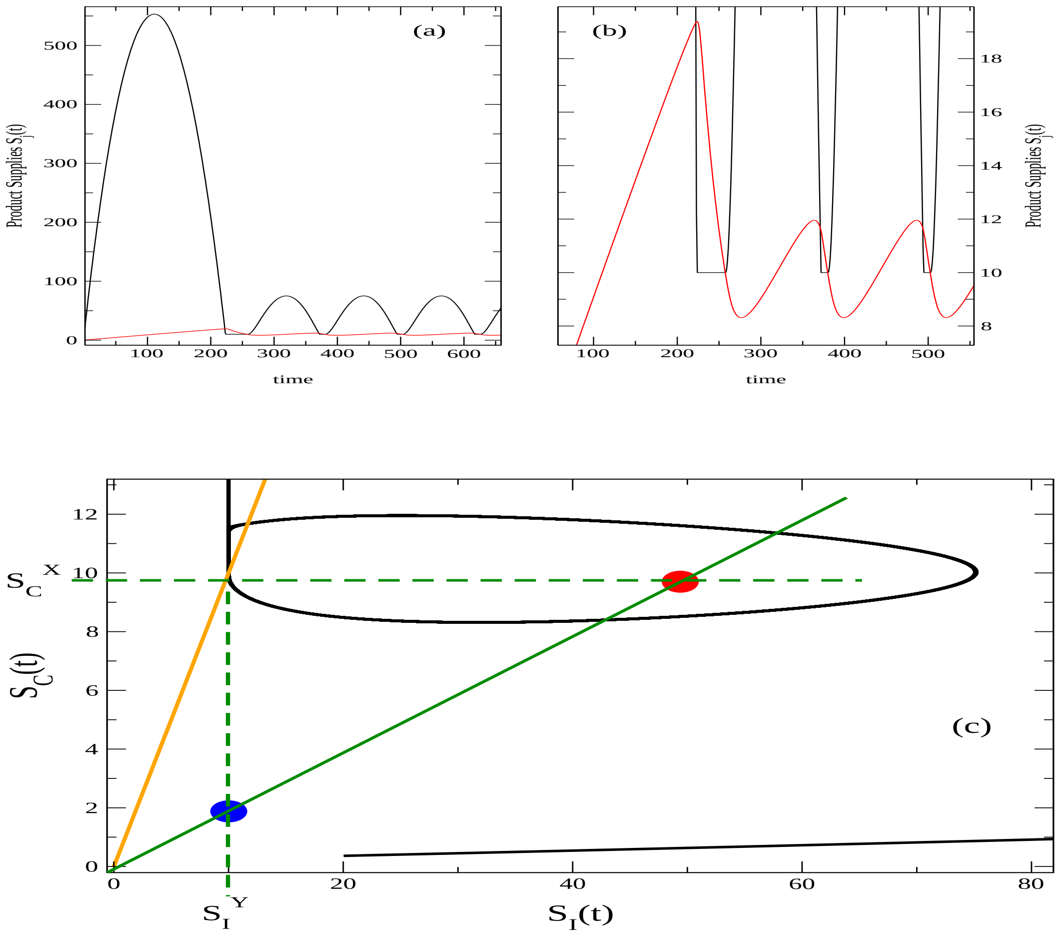

Fig. 4. (a) Single process supplies time series with detail in (b). $S_I(t)$ black, $S_C(t)$ red. (c) The same time series in the $S_I(t)$, $S_C(t)$ plane. The origin line is the $S_I(t) = S_C(t)$ line and divides the plane into the (X) and (Y) regions. When the trajectory crosses this line the equations *switch* the minimum condition. The (Y) fixed point (S_I^Y, S_C^Y) is shown blue and the (X) fixed point (S_I^X, S_C^X) is shown red. They lie on the green line of slope ρ through the origin. The parameters are $S_I^{ext} = 10$, $D_O^{ext} = 100$, $S_C^{ext} = 0.1$, $D_C^{ext} = 1.5$ and $D_I^{ext} = S_O^{ext} = 0$. $\gamma = 0$, $\alpha = 1$, $a_I = a_C = b_C = b_O = 1/2$.

and it is clear that the external demands, D_O^{ext} etc, are also to be considered *rates* of supply of money from the external environment. Indeed the process receives all the available value at these rates since it is the sole owner of these supplies and has no competition to supply them.

We now turn to the more general case decribed by Eqs.14, where none of the external supplies and demands are necessarily zero.

The time series behaviour of the supplies $S_I(t)$ and $S_C(t)$ in this case is shown in Fig.4. The initial part, up to $t \sim 270$m, is a transient and the roughly quadratic behaviour of $S_I(t)$ and linear growth of $S_C(t)$, as described for the simplified case above, is easy to see. It is also seen that once $S_I(t)$ has crossed

$S_C(t)$ at $t \sim 220$ the input supply goes quickly to a fixed level as it did in the simple case described above, but rather than $S_C(t)$ continuing to grow linearly it now decays. After some time $S_C(t)$ crosses back through $S_I(t)$ and the system switches back to the roughly quadratic $S_I(t)$ growth behaviour. This state is repeated to produce a *permanent switching* state, which is a novel type of limit cycle.

Since the nature of this oscillation is relevant for the understanding of business cycles, even in this $\gamma = 0$ no price feedback case, we now consider it in some detail.

Indeed the origin of this limit cycle is quite novel in dynamical systems theory since it is produced basically by the presence of the minimum condition which switches between the catalyst minimum phase and the input minimum phase. We now denote these phases (X), $S_I(t) > S_C(t)$ and (Y) $S_C(t) > S_I(t)$ respectively. This limit cycle can be understood simply by considering the fixed points of the system Eq.14, (where we now set all the non-zero stochiometric ratios to 0.5.) In both (X) and (Y) phases the system has two fixed points. In both cases (X) and (Y) one of the fixed points is at the origin $(S_I = 0, S_C = 0, F = 0)$, (we will say more about this later.) The other fixed point is at a different position in each phase however. Denoting the (X) catalyst minimum phase phase fixed point as (S_I^X, S_C^X, F^X) and the (Y) input minimum phase fixed point as (S_I^Y, S_C^Y, F^Y) we find,

$$R^{equil} = S_C^X = S_I^Y = \frac{S_I^{ext} D_O^{ext} - S_O^{ext} D_I^{ext}}{D_O^{ext} + D_I^{ext}} = S_O^{X,Y} \tag{16}$$

$$\rho = \frac{S_I^X}{S_C^X} = \frac{S_I^Y}{S_C^Y} = \frac{D_C^{ext}(S_I^{ext} + S_O^{ext})}{S_C^{ext}(D_I^{ext} + D_O^{ext})} \tag{17}$$

and

$$F^X = 2S_C^X \frac{D_C^{ext}}{S_C^{ext}} \qquad F^Y = 2S_C^Y \frac{D_C^{ext}}{S_C^{ext}} \tag{18}$$

The details of these fixed points are not important, we include them for completeness, except that the ratio of the fixed point values which we denote ρ, is the same in both cases and that $S_C^X = S_I^Y$. This geometry means the fixed points lie on a single line of slope ρ through the origin as shown in Fig.4. The important point to notice is that at the parameter settings used for this figure while the (X) catalyst minimum fixed point is in the (X) phase, the (Y) fixed point is not in the (Y) phase. This means that if the trajectory enters the (Y) phase it will be attracted to a fixed point in the (X) phase causing the system to switch back into the (X) phase some time later. The other important point to notice is that the (X) phase has an oscillatory character. In fact this fixed point is a focus, while the (Y) fixed point is a node. The trajectory transiently oscillates around this fixed point, but before it can decay to it, it crosses back into the (Y) phase producing the novel limit cycle. Note also that if the external environment parameters are such that $\rho < 1$, the system can remain at the node (Y) fixed point without oscillating. In this case $S_I(t)$ and $S_C(t)$ decay to fixed values, with $S_C(t)$ in excess - a state of permanent unemployment. $\rho = 1$ therefore produces a bifurcation from fixed point to limit cycle.

While the switching condition turns a transient oscillation into the a limit cycle, it is not the origin of the oscillation itself. This is caused by the focus in the (X) phase. It is economically interesting to understand the origin of this oscillation and to do this we consider the simplest oscillating system obtained by setting the external demand for input $D_I^{ext} = 0$ and the external supply of the output $S_O^{ext} = 0$. This means the process has no competition for the supply of input and therefore receives it all. Furthermore it has no competition to supply output and so receives all the money, D_O^{ext}, for the output regardless of how much it can supply. The equations Eq.14 now become, (after some reorganisation).

$$\frac{dS_I(t)}{dt} = S_I^{ext} - Min(S_C, S_I) \tag{19}$$

$$\frac{dS_C(t)}{dt} = \frac{1/2F(t)S_C^{ext} - S_C(t)D_C^{ext}}{1/2F(t) + D_C^{ext}} \tag{20}$$

$$\frac{dF(t)}{dt} = -\frac{1/2F(t)S_I^{ext}}{S_I(t) + S_I^{ext}} + \frac{S_C(t)D_C^{ext} - 1/2F(t)S_C^{ext}}{S_C(t) + S_C^{ext}} + D_O^{ext} \tag{21}$$

where the 1/2 factors are the stochiometric ratios which we have fixed to 1/2 and the fixed points, Eq.16, reduce to $S_C^X = S_I^Y = S_I^{ext}$.

When the trajectory enters the (Y) $S_I(t)$ minimum phase, as can be seen from Eq.19, the input supply equation decouples from the rest of the system and $S_I(t)$ decays immediately to $S_I(t) \sim S_I^{ext}$. Therefore the factor in the first term in Eq.21 becomes constant and the system basically decays without oscillating according to the Eq.20 and the second term in Eq.21. Notice that the $S_C(t)$ fixed point is such that $1/2F^Y S_C^{ext} = S_C^Y D_C^{ext}$. This simply states that the rate of catalyst received by the process from its demand i.e. $1/2F^Y S_C^{ext}$ is equal to the rate the catalyst is lost to external demand $S_C^Y D_C^{ext}$. As explained this fixed state is impossible to achieve in the (Y) phase if the external parameters are set as described in Fig.4 so that $\rho > 1$. Before the system has decayed to this fixed point it will switch back into the (X) phase. Here the $S_I(t)$ equation Eq.19 now has an $S_C(t)$ term and it doesn't decouple from the rest of the system and the system oscillates.

We will study this oscillation in more detail when we consider the price feedback case $\gamma \sim 1$ below, but for now we mention three points.

(i) We see that the catalyst oscillates around its fixed point level, $S_C^X = S_I^Y$. Indeed this is also seen to be the level where input and catalyst supplies are correctly balanced in their correct stochiometric ratios. In this simplified system, since $D_I^{ext} = 0$, the process receives all the external input S_I^{ext} no matter how much it demands and $S_C^X = S_I^{ext}$ is indeed the correctly balanced desired catalyst level. In Fig.4 this value is set to 10. Therefore the process does indeed try to continuously adjust its catalyst supply to balance the external input supply rate S_I^{ext}. This is an interesting non-trivial emergent result, since we nowhere 'told' the model to increase the catalyst when the input supply rate was in excess, or vice versa. It is simply a consequence of the competetive value $F(t)$ dynamics. Furthermore, as we shall see, it also holds when we set the price feedback γ to non-zero levels.

We also note that the expression $S_C^X = S_I^Y$ is also the equilibrium production $R^{equil} = S_O^{X,Y}$ defined by Eqn.12 since in the X phase $S_C(t)$ is the production rate and in the Y phase $S_I(t)$ is the production rate. I.e. this is the fixed point of the minimum, which is the same in both phases. The catalyst oscillates around the equilibrium production rate R^{equil} determined by the external supply rate $R^{equil} = S_I^{ext}$.

The oscillation itself however changes the processing rate in Eq.19. There are periods where there is insufficient catalyst for processing the external supply rate S_I^{ext}. This results in the quadratic oscillatory behaviour of $S_I(t)$ as described above.

(ii) The $S_C(t)$ oscillation is a kind of *threshold oscillation*. Indeed in Eq.20, when $F(t)$ is large so that $F(t)a_C >> D_C^{ext}$, so that this production process is much more valuable than the external demand, we basically recover linear growth $S_C(t) \sim S_C^{ext}t$ as the process takes all the external supply. This linear growth in $S_C(t)$ is the maximum attainable given the fixed external supply rate S_C^{ext}. The linear growth of $S_C(t)$ increases the processing rate in Eq.19, whereas in the (Y) case the variation of $S_C(t)$ had no effect on the processing rate. Eventually the $S_C(t)$ level crosses S_I^{ext} and $S_I(t)$ starts to decrease again and will eventually cause a switch back to the (Y) phase. When $F(t)a_C << D_C^{ext}$, however, the catalyst supply $S_C(t)$ in Eq.20 basically follows $S_C(t) \sim F(t)a_C S_C^{ext}/D_C^{ext}$ which has small value. Indeed the switch over between these two regimes can be very sudden and catastrophic and we have an oscillation of roughly linear catalyst growth followed by sudden catalyst collapse. This typical dynamics is shown in Fig.4(b), and we will go into it in more detail when we consider the $\gamma \sim 1$ case below.

(iii) The bifurcation parameter ρ also controls the frequency of the oscillation. This is due to the interaction of the line of slope ρ which contains the fixed

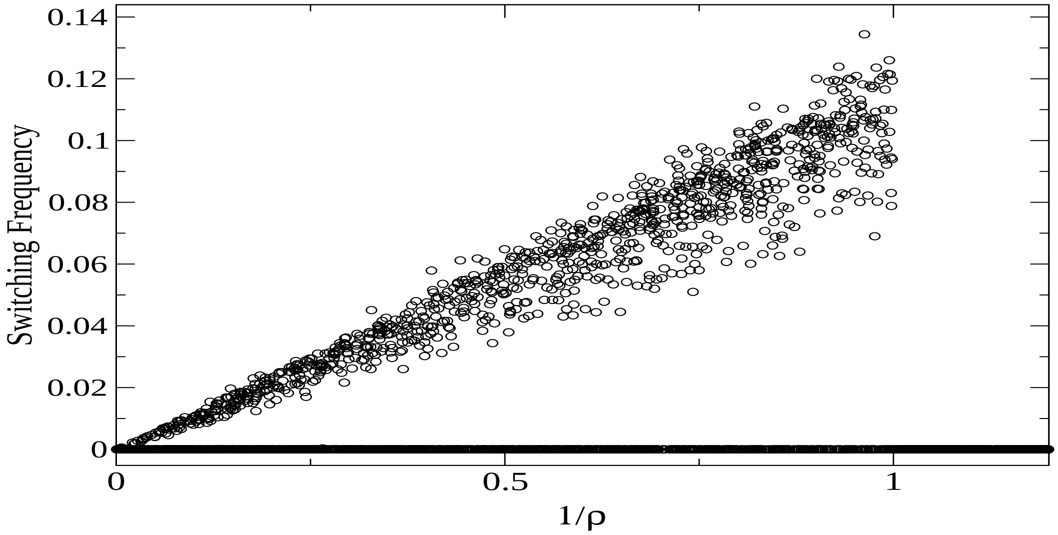

Fig. 5. Variation of swithching frequency with $1/\rho$ for single process with $\gamma = 0$. Each point is the result of a single simulation with the 6 external environment supplies and demands chosen randomly and ρ thereby calculated according to Eq.17. The roughly linear behaviour with $1/\rho$ is evident as is the bifurcation to a non-oscillating state at $\rho = 1$.

points and the minimum switching line $S_I = S_C$. This can be seen from Fig.4(c). As the parameter ρ decreases towards the bifurcation point 1, the frequency of the oscillation will increase and the amplitude of the oscillation will decrease since the focus fixed point will approach the switching boundary. This is confirmed in Fig.5 which shows the switching frequency as ρ is varied. The behaviour is rougly linear in $1/rho$ as would be expected from geometrical considerations.

3.1.2 Business cycle oscillation with price feedback

Indeed one may wonder why we have gone through this detailed analysis for the case where there is no price feedback, i.e. $\gamma = 0$. Surely the process could balance its input and catalyst supplies better by allocating its funds $F(t)$ more efficiently than simply according to the stochiometric ratios, without even looking at the possibly very different product prices? However this is does not seem to be the case. When the process uses price information to calculate its fund allocation, i.e. its demand ratios σ_{ij}, we recover basically the same behaviour.

Fig.6(a) shows supplies time series for the case when $\gamma = 0.99$ in Eq.10. Again we describe the simplest oscillating case where $D_I^{ext} = S_O^{ext}$ and where all the stochiometric ratios are 1/2. Furthermore when considering price feedback, for simplicity, we consider the discrete time version of Eq.7-10 and in Eq.10 set $\tau = 1$. That is, the process uses the last known available price to allocate

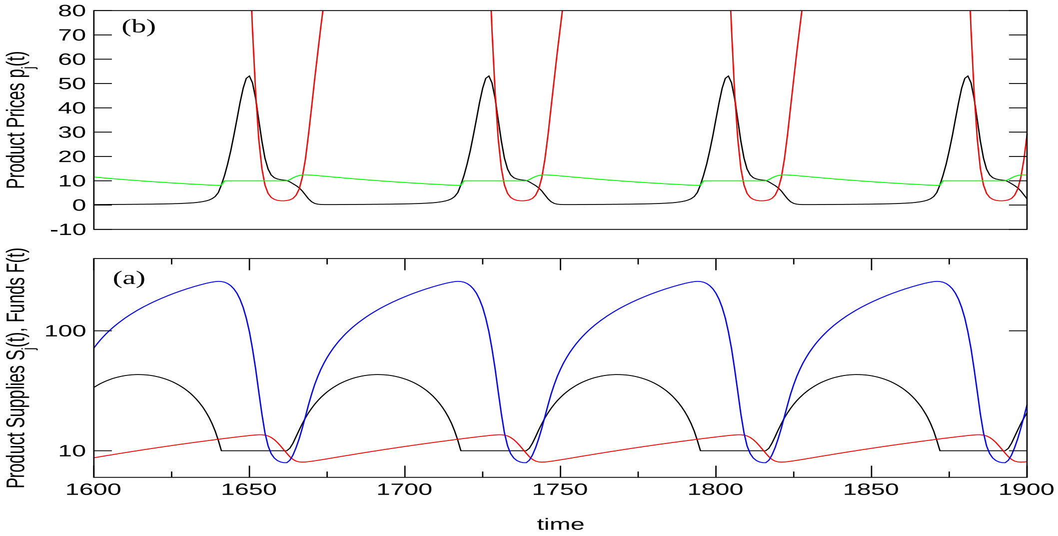

Fig. 6. (a) Single process supplies time series, $S_I(t)$ black, $S_C(t)$ red. Also shown is the funds $F(t)$, blue, rescaled by S_C^{ext}/D_C^{ext}. The parameters are the same as in Fig.4, except $\gamma = 0.99$. (b) Corresponding time series of product prices, $p_I(t)$ black, $p_C(t)$ red, $p_O(t)$ green.

its money, in a similar way to how such fund allocation would be decided in reality.

As can be seen, we find the same threshold behaviour, more dramatically now, with sudden collapses in catalyst supply separating longer periods of linear growth. The reason is basically closely related to the above, but now we show how it is intimately connected with the behaviour of the prices. The corresponding behaviour of the prices is shown in Fig.6(b). Notice that when the system is in the catalyst minimum state the input supply is in excess and its price is almost fixed at zero. Indeed this is precisely what one knows and expects from reality, but this fact has emerged as a result of our model rather than being an assumption as it is in the VNM in Eq.4. Indeed since D_I^{ext} is set to zero here, the process has no competition for the input supply S_I^{ext} and therefore receives it all, no matter what the price $p_I(t)$, and indeed can 'dictate' the price. This dictated price is simply $p_I(t) = F(t)\sigma_I(t)/(S_I(t) + S_I^{ext})$. In the previous case we had $\sigma_I = a_I$ and therefore the process always used money buying the input $S_I(t)$. However when we include price feedback we find the interesting and unexpected result that the process reduces its allocated funds $\sigma_I(t) \sim 0$ to zero, exactly as would occur in reality, and uses all its funds to demand the catalyst which is the product in insufficient supply, so that $\sigma_C(t) \sim 1$. The input supply price $p_I(t)$ accordingly drops to zero while the catalyst price $p_C(t)$ bubbles to a high level. This is only true however in the (X) case that the process has excess input $S_I(t) > S_C(t)$. As can be seen from Fig.6 when the process has excess catalyst, (Y) the process reverses its demands $\sigma(t)$ allocation so that $\sigma_C(t) \sim 0$ and $\sigma_I(t) \sim 1$ so that the price of the catalyst collapses to its base level $p_C(t) = (F(t)\sigma_C(t) + D_C^{ext})/(S_C(t) + S_C^{ext}) \sim$

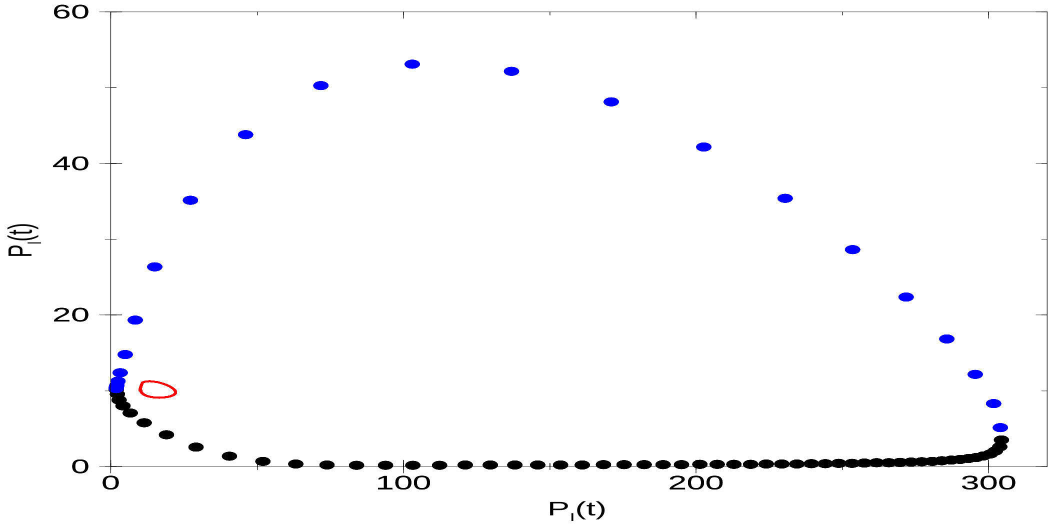

Fig. 7. Emergent dynamical rationality. The red line shows the variation of catalyst price $p_C(t)$ and input price $p_I(t)$ in the $\gamma = 0$ case in the $p_C(t), p_I(t)$ plane. The other time series is the $\gamma = 0.99$ case. The points are shown black when the input supply is in excess and blue when the catalyst supply is in excess. As can be seen when a supply is in excess its price is near zero. This result expected from rational consideration only occurs when $\gamma \sim 1$.

$D_C^{ext}/(S_C(t) + S_C^{ext})$ while the price of the input $p_I(t)$ bubbles due to the funds reallocated. That is we find the interesting result the the price of the rate determining quantity, i.e. the minimum quantity, in an economic production process, will bubble, while the price of products in excess will collapse.

Returning to the supplies time series we see that the catalyst executes threshold oscillations around the input supply level S_I^{ext}. This is again, as described above for the $\gamma = 0$ case the correct level for balanced inputs. Indeed in the (X) catalyst minimum phase the process uses all its funds $F(t)$ to demand catalyst. Nevertheless the catalyst can only grow at the maximum linear rate dictated by the external environment fixed supply rate S_C^{ext}. Eventually this increases the processing rate and the input supply $S_I(t)$ decreases through the catalyst supply. At this point the process has lots of catalyst and is processing all its input at the external supply rate S_I^{ext}. But what happens now is quite revealing. Since the process no longer requires catalyst, as explained, it doesn't demand it anymore and the catalyst price collapses. Since the process has lots of catalyst, and the price is collapsing, now suddenly the process' value $F(t)$, also shown in Fig.6 collapses. Indeed now the process is almost worthless. As described above for the threshold behaviour of the catalyst, since now $F(t) << D_C^{ext}$, the catalyst supply, in turn, suddenly collapses. And the cycle of growth and collapse, as explicitly seen in the time series of the value $F(t)$, continues.

Indeed, very approximately, the situation is easy to understand mathematically in terms of a slow and fast manifold. For the single process in discrete time the equations can be written,

$$\begin{aligned}
S_I(t+1) &= \frac{\sigma_I(t)F(t)}{p_I(t)} \\
S_C(t+1) &= \frac{\sigma_C(t)F(t)}{p_C(t)} \\
F(t+1) &= (S_I(t) - M(S_I(t), S_C(t)))p_I(t) + S_C(t)p_C(t) + M(S_I(t), S_C(t))p_O(t)
\end{aligned}$$

where $M(x)$ now and in the following is shorthand for the production function, i.e. the minimum condition, and the demands for input and catalyst are $\sigma_{I,C}$,

$$\sigma_I(t) = 1 - \sigma_C(t) = \gamma\frac{p_I(t-1)}{p_I(t-1) + p_C(t-1)} + (1-\gamma)1/2$$

and in the simplest oscillating process which has no competition when buying input so that $D_I^{ext} = 0$ and where $\gamma \sim 1$ the prices are given by,

$$\begin{aligned}
p_I(t) &= \frac{\sigma_I(t)F(t)}{S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext}} \sim \frac{p_I(t-1)}{p_I(t-1) + p_C(t-1)}a(t) \\
p_C(t) &= \frac{\sigma_C(t)F(t) + D_C^{ext}}{S_C(t) + S_C^{ext}} \sim \frac{p_C(t-1)}{p_I(t-1) + p_C(t-1)}b(t) + c(t)
\end{aligned}$$

In the region where $F(t) >> D_C^{ext}$ we can neglect $c(t)$ and then considering $a(t)$ and $b(t)$ slowly varying the prices have two “fixed points”, i.e. $p_I^* = 0, p_C^* = b(t)$ and $p_I^* = a(t), p_C^* \sim 0$, the stability of which depends on the ratio $a(t)/b(t) > 1$, which is simply given by,

$$\frac{a(t)}{b(t)} \sim \frac{S_C(t) + S_C^{ext}}{S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext}} \tag{22}$$

Therefore as processing occurs at rate $M(S_I(t), S_C(t))$ the stabilty of the price fixed points can suddenly switch causing a switchover in the price fixed points - i.e. bubbles and crashes. Since just before the minimum condition $M(x)$ switches $S_I(t) - M(S_I(t), S_C(t))$ is of the order of size of $S_C(t)$ while after it $S_I(t) - M(S_I(t), S_C(t))$ is zero the price attractor switch will most likely occur when the the minimum $M(x)$ switches if S_I^{ext} is “small”.

Therefore in this case considered here where there is no external demand for input so that $D_I^{ext} = 0$ the input price will be attracted to zero when $S_I(t) - M(S_I(t), S_C(t))$ is large i.e. when there is excess input supply. This is exactly what one would expect when considering “rational agents” since in the case where there is no competition to buy the input, the process can "dictate" the input price and should therefore get it for free. This is described in Fig.**??**. As described in the introduction this is a postulate in the equilibrium analysis of the VNM, but here has appeared as a dynamical consequence, but only in the case with price feedback $\gamma \sim 1$ i.e. some rationality.

Indeed in the state where input price is near zero $p_I(t) \sim 0$ and $\sigma_I(t) \sim 0, \sigma_C(t) \sim 1$ we find,

$$\begin{aligned}
S_I(t+1) &\sim S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext} \\
S_C(t+1) &\sim (S_C(t) + S_C^{ext}) \frac{F(t)}{F(t) + D_C^{ext}} \\
F(t+1) &\sim (F(t) + D_C^{ext}) \frac{S_C(t)}{S_C(t) + S_C^{ext}} + \frac{M(S_I(t), S_C(t))}{M(S_I(t), S_C(t)) + S_O^{ext}} D_O^{ext}
\end{aligned}$$

This state describes the funds $F(t)$ and catalyst supply $S_C(t)$ growing slowly while the input supply $S_I(t)$ gradually decreases as the production rate $M(S_I(t), S_C(t)) = S_C(t)$ increases. Eventually the price fixed points stability changes and and a fast attractor switch occurs to a new state where the input price $p_I(t)$ suddenly "bubbles" and the catalyst price collapses $p_C(t) \sim 0$. Then $\sigma_C(t) \sim 0, \sigma_I(t) \sim 1$ and we get,

$$\begin{aligned}
S_I(t+1) &\sim S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext} = S_I^{ext} \\
S_C(t+1) &\sim 0 \\
F(t+1) &\sim F(t) \frac{S_I(t) - M(S_I(t), S_C(t))}{S_I(t) - M(S_I(t), S_C(t)) + S_I^{ext}} + \frac{M(S_I(t), S_C(t))}{M(S_I(t), S_C(t)) + S_O^{ext}} D_O^{ext} \\
&\sim \frac{S_I(t)}{S_I(t) + S_O^{ext}} D_O^{ext}
\end{aligned}$$

As can be seen from these equations now the catalyst supply $S_C(t)$ quickly collapses due to the switch in demands for input and catalyst. This sudden collapse in $S_C(t)$ affects the price stability ratio $a(t)/b(t)$ Eq.22 and causes a sudden switch back to the zero input price phase. The effect is to produce a very fast bubble in the input price rapidly followed by a sudden crash in the input price. In the extreme case of $\gamma = 1$ shown in Fig.8 this bubble resembles a delta function.

These results are confirmed in Fig.7 which shows the time series variation in the input product price $p_I(t)$, catalyst product price $p_C(t)$ plane. The price of the input product spends a long time near zero when it is in excess supply and similarly for the catalyst price. This can be seen as evidence of "emergent dynamical rationality" when $\gamma \sim 1$ and the process can "observe" the previous time, last known price. This rational behaviour does not occur when $\gamma = 0$. The slow-fast switching dynamics is also evident.

It seems that such cyclic bubble and crash behaviour is unavoidable in economics due to the constraints of the fixed external supply and demand rates, $S^{ext}_{I,C,O}$ and $D^{ext}_{I,C,O}$, which limit whatever growth rates are possible, no matter how much money the process has available, coupled with the fundamental fact that the process itself acts on the prices through its demands and therefore to a certain extent affects its own value in a malicious way. That is, when a production process has an excess of something it will not require it and this will depress its price. This will depress the value of the process dramatically since by definition it affects the thing the process has most of. As shown this can happen in a catastrophic way due to threshold nature of economic dynamics.

3.1.3 Price Equilbrium Behaviour

We now briefly return to the simpler no price feedback $\gamma = 0$ case to consider the average price levels. If we substitute the fixed point values in the two phases (X) and (Y) Eqs.16,17,18 into the price equations Eq.9 we get the equilibrium fixed point prices. These turn out to be the same in both phases (X) and (Y) and are given by,

$$p_I^{equil} = p_O^{equil} = \frac{D_I^{ext} + D_O^{ext}}{S_I^{ext} + S_O^{ext}} \qquad p_C^{equil} = \frac{D_C^{ext}}{S_C^{ext}} \tag{23}$$

where p_I^{equil}, p_O^{equil} and p_C^{equil} are the equilibrium prices of the input, output and catalyst respectively.

Above we also mentioned that each phase has another fixed point, at the origin, the *bankrupt* process fixed point. In this case, or when the process doesnt exist, the equilibrium fixed point prices are simply given by,

$$p_I^{brupt} = \frac{D_I^{ext}}{S_I^{ext}} \qquad p_O^{brupt} = \frac{D_O^{ext}}{S_O^{ext}} \qquad p_C^{brupt} = \frac{D_C^{ext}}{S_C^{ext}} \tag{24}$$

where ^{brupt} refers to bankrupt fixed point prices. In fact the interior fixed point, Eqs.16,17,18,23, is stable, and the bankrupt fixed point unstable, when $\frac{D_I^{ext}}{S_I^{ext}} < \frac{D_O^{ext}}{S_O^{ext}}$, i,e. when $p_I^{brupt} < p_O^{brupt}$. If this is the case the process can exist without going bankrupt and it does so by processing input product into output product at a rate such that the equilibrium prices given by Eq.23 hold and the equilibrium price of the input is now equal to the output. Of course the system, as explained above, is not necessarily at a fixed point, but nevertheless will oscillate around it, so the relations Eq.23 approximately hold on average. In the case that $p_I^{brupt} > p_O^{brupt}$ the process ‘goes bankrupt’ and does not affect the prices. Indeed the bifurcation to bankruptcy occurs smoothly. As noted the process equilibrium production Eq.12 is given by $R^{equil} = S_C^X = S_I^Y$,

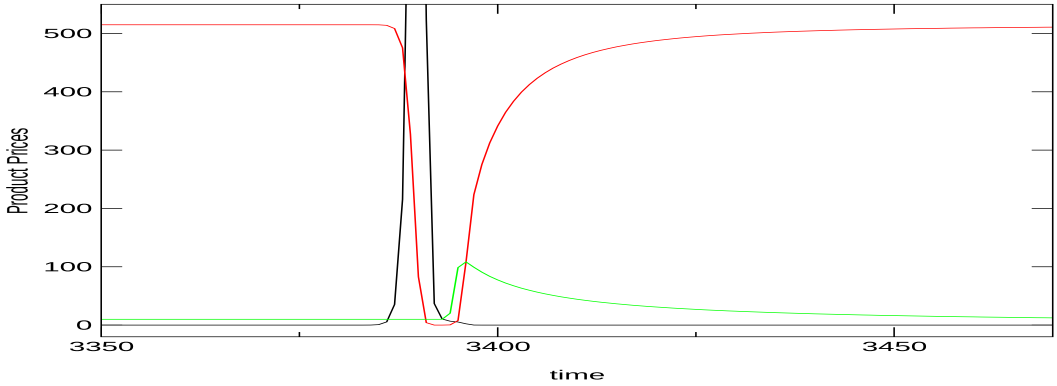

Fig. 8. Time series of product prices, $p_I(t)$ black, $p_C(t)$ red, $p_O(t)$ green. Parameters same as in Fig.6 except $\gamma = 1$.

Eq.16. As we approach the bankruptcy bifurcation the numerator in this R^{equil} expression, $S_I^{ext} D_O^{ext} - S_O^{ext} D_I^{ext}$ goes smoothly to zero and the production drops to zero. In the VNM equilibrium model this zero production condition is included as Eq.2. In our model, again, this behaviour naturally emerges as a dynamical result when the process is unprofitable.

Furthermore at equilibrium the relation Eq.23 says that both the input price and output price are equal to the total external demand divided by the total external supply. This non-trivial relationship means that for example changing the external supply of the output S_O^{ext} will not only affect the output product price, but also the input product price. This is because the equilibrium processing rate, $S_C^X = S_I^Y$, is affected by changing S_O^{ext} which feeds back onto the input product price. This is of course providing the quantities aren't changed so much that system goes bankrupt.

A surprising thing is that this relationship also seems to hold for price feedback $\gamma \sim 1$ case. This is observed in Fig.6(b). In this example $D_I^{ext} = S_O^{ext} = 0$, $D_O^{ext} = 100$ and $S_I^{ext} = 10$ so that $p_I^{equil} = p_O^{equil} = 10$. As can be seen $p_O(t)$ clearly moves around 10, while $p_I(t)$ may also average around this value but with periods when it is zero and periods when it bubbles strongly. It is interesting that the input price bubbles are only weakly transferred to the output price. Indeed in a real economy some sectors may be bubbling while others are not. The catalyst equilbrium price p_C^{equil} in Eq.23 is not obeyed when $\gamma \neq 0$, however, as can be seen from the time series.

In Fig. 8 we show the situation of perfect price feedback $\gamma = 1$. Now the price changes are even more dramatic. In the stable periods of excess input $S_I(t) > S_C(t)$ we find the input price fixed at zero, the output price fixed at 10 and the catalyst price fixed at a high value. At time $t = 3350$ noone would suspect the catastrophic price swings that will occur when the catalyst supply $S_C(t)$, growing linearly, goes into excess and becomes suddenly worthless.

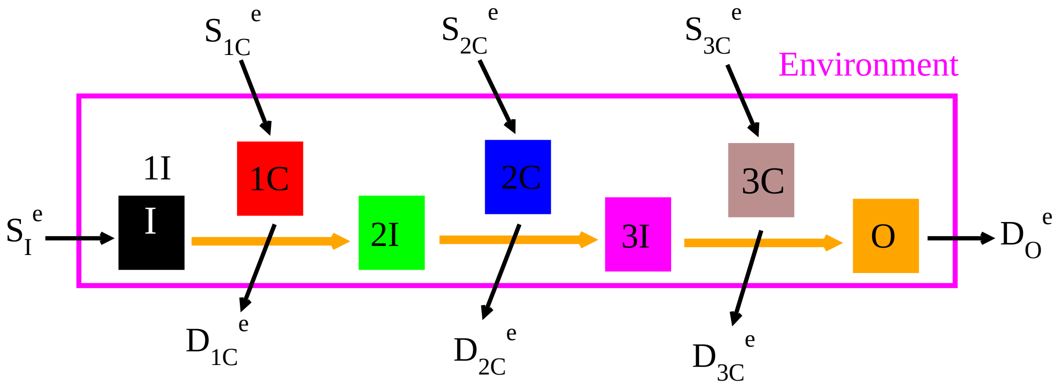

Fig. 9. Diagram of a 3 process chain with 7 products in fixed environment. 3 products are catalysts, one in each process each with a fixed external supply S_{iC}^{ext} and demand D_{iC}^{ext}, defining an external price for each catalyst. There is one input product I with a fixed external supply S_I^{ext} and one output product O with fixed external demand D_O^{ext}. There are two intermediate products with no external supplies and demands, but which are supplied by the feeding process and demanded by the drawing process.

3.2 Chain of Processes

We now come to consider what happens when multiple processes are coupled together. We show that in general we expect complex *multiple-timescale* dynamics to appear when multiple processes are coupled.

The simplest and most economically relevant system is the chain of processes in a fixed environment. Such a system might describe a factory production line. We show that this system has a novel multiple timescale limit cycle attractor and that accordingly we expect the output production to be dynamically rather complex.

Such a 3 process 7 product chain is shown in Fig.9, while the a detail from the supplies time series for this system is shown in Fig.10. The time series colours refer to the same colours as in Fig.9. The time series shows a segment from a multiple timescale limit cycle attractor.

It is easy to understand how this behaviour is produced from the previous single process example. From Fig.4 we can see that as the slope of the line, ρ, containing the fixed points Eq.16, increases, the amplitude and therefore period of the limit cycle increases. If the situation is such that each process has a different ρ parameter, then each process will have a different intrinsic oscillation frequency. We also observe that the parameter ρ is given by $\rho = p_C^{equil}/p_I^{equil}$, in terms of the equilibrium prices Eq.23. For this 3 process system these quantities are given by,

$$p_{1C}^{equil} = \frac{D_{1C}^{ext}}{S_{1C}^{ext}} \qquad p_{2C}^{equil} = \frac{D_{2C}^{ext}}{S_{2C}^{ext}} \qquad p_{3C}^{equil} = \frac{D_{3C}^{ext}}{S_{3C}^{ext}} \tag{25}$$

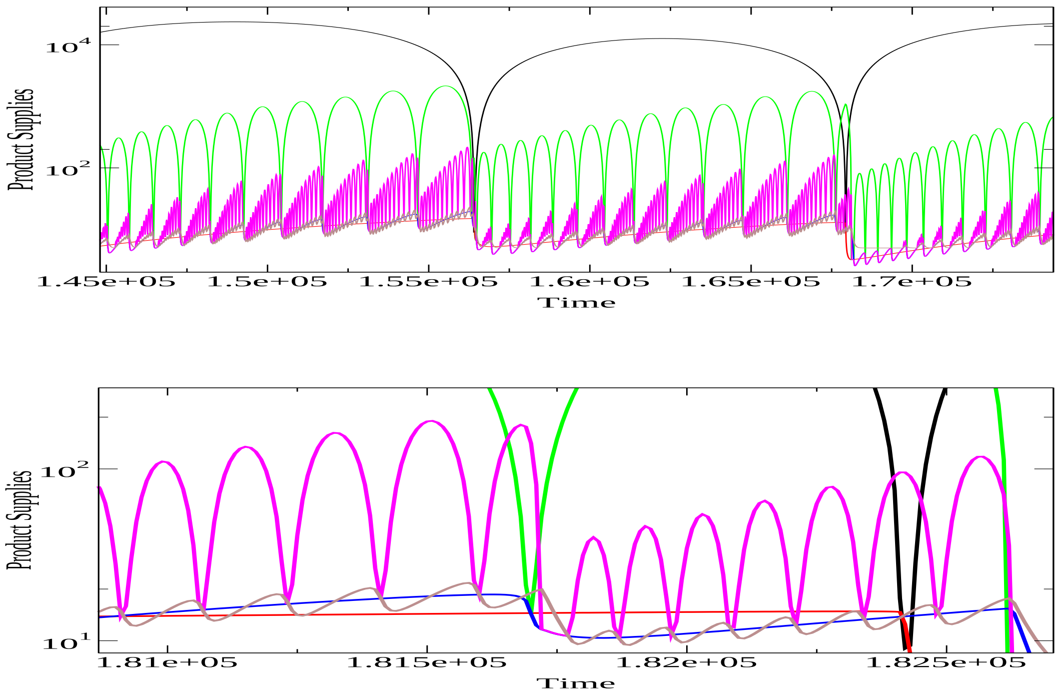

Fig. 10. The 6 time supplies time series for the 3 process chain described in the text. Black: $S_{1I}(t)$, red: $S_{1C}(t)$, green: S_{2I}, blue: $S_{2C}(t)$, pink: $S_{3I}(t)$, brown: $S_{3C}(t)$. Lower panel is detail of top panel. No price feedback $\gamma = 0$ case.

for the catalyst prices. While the input and output fixed point prices are,

$$p_I^{equil} = p_O^{equil} = \frac{D_O^{ext}}{S_I^{ext}} \tag{26}$$

where we get this last relation, Eq.26, from the fact that because of the input and output price equilization considerations above, chains of processes must all have the same input and output equilibrium prices. Of course the system is not at equilibrium, but nevertheless the input and output prices in a chain should move approximately around the same long term average levels.

Now if we set the adjust the external environment such that for example,

$$p_{1C}^{equil} >> p_{2C}^{equil} >> p_{3C}^{equil} > p_I^{equil} \tag{27}$$

then all processes oscillate with very different frequencies. The first process with the highest ρ value oscillates slowest the next at a higher frequency and so on. In this example the catalyst equilibrium price in the first process is 10 times the second which is 10 times the third, which is slightly greater than p_I^{equil}. This situation could of course occur quite easily in reality. These 3 different trapped frequencies are easy to see in Fig.10, where black and red switch on the slowest frequency, green and blue at intermediate switching frequency and pink and brown on the highest frequency.

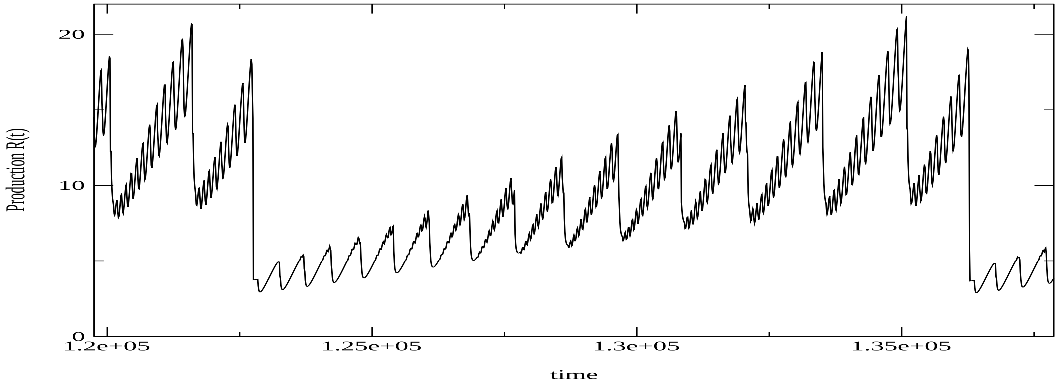

Fig. 11. Output production $R(t)$ time series from the 3 process chain described in the text.

In this case according to Eq.16 since the external demand for the input D_I^{ext} is fixed at zero for convenience in all 3 of the processes (although the same dynamics appear when the intermediate products also have external demands and supplies) we get that the catalyst supply fixed point for process 1 is simply given by $S_{1C}^X = S_I^{ext}$ I.e the catalyst will oscillate around the external supply level as the process tries to keep the catalyst and the external supply level the same. Similarly for the second process catalyst fixed point we get, $S_{2C}^X(t) = S_{1O}(t) = S_{1C}(t)$ (where now the 'fixed point' is slowly time varying) so that the second process catalyst supply will oscillate around the input supply which is the output supply from process 1 which is of course also the catalyst supply for process 1 since the catalyst is the minimum. Similarly we get $S_{3C}^X(t) = S_{2O}(t) = S_{2C}(t)$ for process 3 and this explains why the catalyst levels, the red, blue and brown all seem to get 'sewn' onto each other in Fig.10, in a hierarchical cascade.

Indeed that the catalysts should all oscillate around the same level in non-branching chains of processes is simply due to the fact that the fixed point of the minimum supply i.e. R^{equil}, the production rate, must be the same in each process in the chain. In this case where there is no external demand for input, so that $D_I^{ext} = 0$, $R^{equil} = S_I^{ext}$ for each process in the chain.

This dynamamical complexity is therefore reflected in the network production $R(t)$ Eq.12, which is the output supply from the last process in the chain shown in Fig.11. This is given by the output from the final process $R(t) = S_{3C}(t)$, which moves in a complex, apparently unpredictable way, around its fixed point, the input supply rate $R^{equil} = S_I^{ext} = 10$.

As mentioned, chains of supply processes with different price catalysts is a typical structure for a real economy, since most products are produced by a series of companies supplying parts to the next link in the chain. This example illustrates that even in the case of a fixed environment, as is the case here, with fixed input supplies and demands for all products the dynamics can be

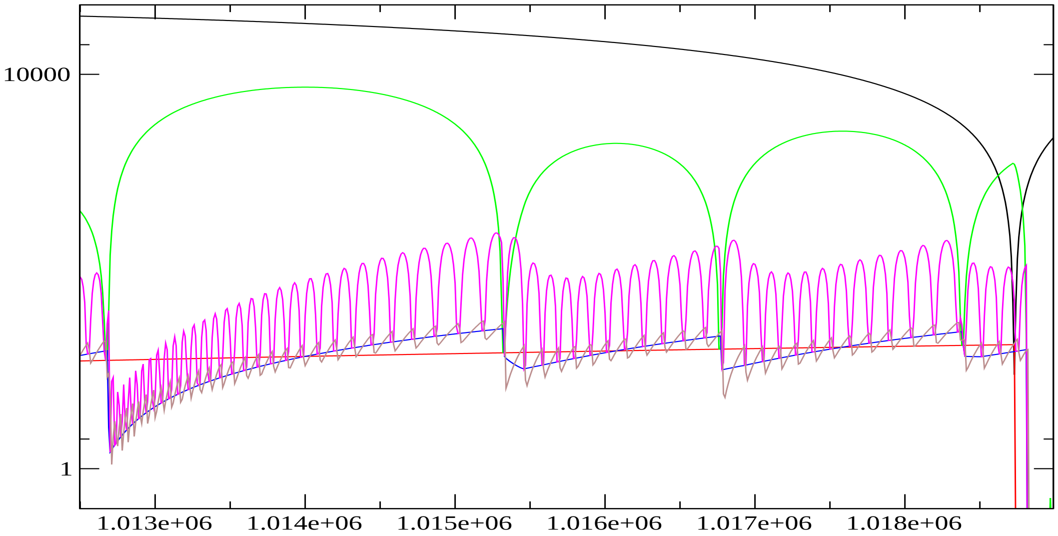

Fig. 12. The 6 time supplies time series for the 3 process chain described in the text. Black: $S_{1I}(t)$, red: $S_{1C}(t)$, green: S_{2I}, blue: $S_{2C}(t)$, pink: $S_{3I}(t)$, brown: $S_{3C}(t)$. In this case there is large price feedback $\gamma = 0.9$.

very complex and difficult to control. Control of such production processes has recently been addressed by Helbing [Helbing (2003)]. We expect such dynamics to therefore be typical of a real economy of coupled production processes.

Again we note that while these analysis and results have referred to the $\gamma = 0$ no price feedback case, increasing price feedback does not remove the complexity of this dynamics and indeed enhances it for the same reasons as explained in the case of the single process. Indeed in Fig.12 we show the individual supplies time series for the same chain but in the case of $\gamma = 0.9$. The same complexity is observed and we strongly suspect this will be the case no matter what prescription the processes use to allocate their demand ratios $\sigma_{ij}(t)$. We conclude that this multiple timescale complexity will be a generic factor of production systems and production networks.

3.3 Cycle of Processes

While economically unrealistic a simple cycle of 3 processes is useful for the illustration of the connection between multiple price levels and multiple timescales. In this example we will show that large differences in price levels between different products leads to the result that oscillations can have large amplitude in some processes but not be visible in others.

This system is, like the chain of processes, easy to understand simply by considering the fixed points. This process cycle is described in Fig.13(c). As shown in this system product D is conserved by the cycle of 3 processes but

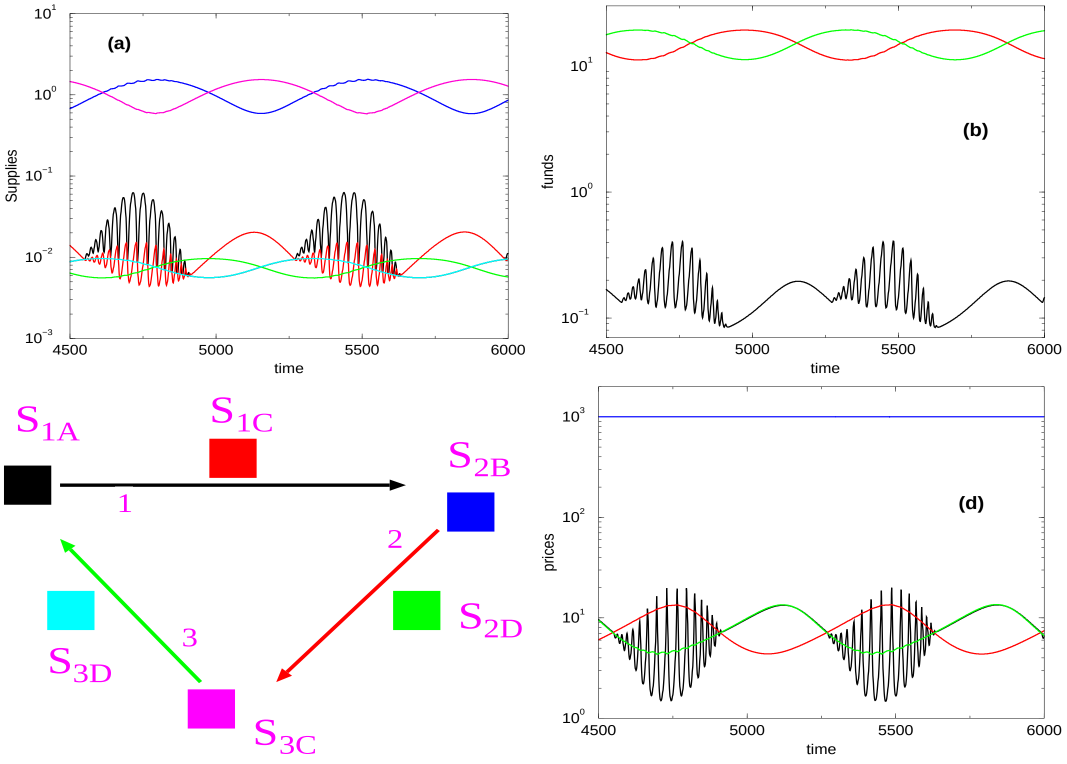

Fig. 13. Time series from process cycle described in bottom left panel (c). There are three processes denoted $1, 2, 3$ and four products A, B, C, D. Process 1 converts A into B via catalyst C. Process 2 converts B into C via catalyst D. Process 3 converts C into A via catalyst D. The colours depict the 6 individual process supplies, whose time series is shown in the top left panel (a). The top right panel (b) shows the 3 funds $F(t)$, process 1, $F_1(t)$, is shown black, process 2, $F_2(t)$, red and process 3, $F_3(t)$, green. The bottom right panel (d) shows the prices, $p_A(t)$ black, $p_B(t)$ red, $p_C(t)$ green, $p_D(t)$ blue.

may be demanded and supplied by the external environment. The total supply of $A+B+C$ is conserved by the system. Since A,B and C are in the same chain they must have same price fixed points. In this example we set the external supply S_D^{ext} and demand D_D^{ext} of product D such that the fixed point price of D, p_D^{equil} is large. We therefore expect,

$$p_A^{equil}(t) \sim p_B^{equil}(t) \sim p_C^{equil}(t) << p_D^{equil}(t) \tag{28}$$

where now however these "fixed point" prices are time varying since the external environment is not fixed.

This is confirmed in Fig.13 (d), where the 3 prices $p_A(t)$, $p_B(t)$ and $p_C(t)$ do move around the same general equilibrium level.

We also therefore get,

$$\rho^1(t) = \frac{p_C^{equil}(t)}{p_A^{equil}(t)} << \rho^2(t) \sim \frac{p_D^{equil}(t)}{p_B^{equil}(t)} = \rho^3(t) = \frac{p_D^{equil}(t)}{p_C^{equil}(t)} \tag{29}$$

and we expect process 1 to have much faster oscillation. This is also apparent from Fig.13. In this example however, unlike the example of the previous 3 process chain, since the equilibrium prices are time varying around the same level we expect the variation of $\rho^1(t)$ to cause a periodic bifurcation in process 1 as it periodically crosses unity. This is clear to see in the figure, where there is a periodic bifurcation from a high frequency oscillating state when $\rho^1(t)$ is slightly greater than 1 to a non-oscillating state where $\rho^1(t)$ is slightly less than 1.

The single processes the funds $F(t)$ fixed points Eq.18 can be seen to be given by,

$$F^{X,Y}(t) = 2R^{equil}p_{min}(t)^{X,Y} \tag{30}$$

That is the funds fixed points $F^{X,Y}$ are different in the two phases X and Y but are always given by the product of the fixed point of the minimum supply R^{equil} which is the same in both X and Y phases and the fixed point price of the minimum product in the two phases, denoted $p_{min}^{X,Y}$ in Eq.30, which is different in the two phases.

Since the 3 processes are in the same chain R^{equil} the fixed point of the minimum supply must be the same for each process. This is clearly seen for the supplies $S_{1A}(t)$, $S_{1C}(t)$, $S_{2D}(t)$ and $S_{3D}(t)$ in Fig.13. Furthermore since $p_D^{equil} >> p_A^{equil}(t) \sim p_C^{equil}(t)$ the funds fixed points must be such that $F_1^{X,Y} << F_2^X \sim F_3^X$. This is clearly seen in Fig.13 (b).

It is this large disparity in funds $F(t)$ between the 3 processes which explains the fact that oscillations are seen in some processes but not others. In particular for example we see from the prices time series that $p_A(t)$ shows the high frequency oscillations from process 1 while $p_B(t)$ and $p_C(t)$ do not, although all three product prices are affected by process 1. In fact for $p_A(t)$ we have,

$$p_A(t) = \frac{S_{1A}(t) + Min(S_{3C}(t), S_{3D}(t))}{1/2F_1(t)} = \frac{S_{1A}(t) + S_{3D}(t)}{1/2F_1(t)} \tag{31}$$

since $S_{1A}(t)$ and $S_{3D}(t)$ are of the same order of size the fast oscillations in $S_{1A}(t)$ and $F_1(t)$ show up in $p_A(t)$. However for $p_C(t)$

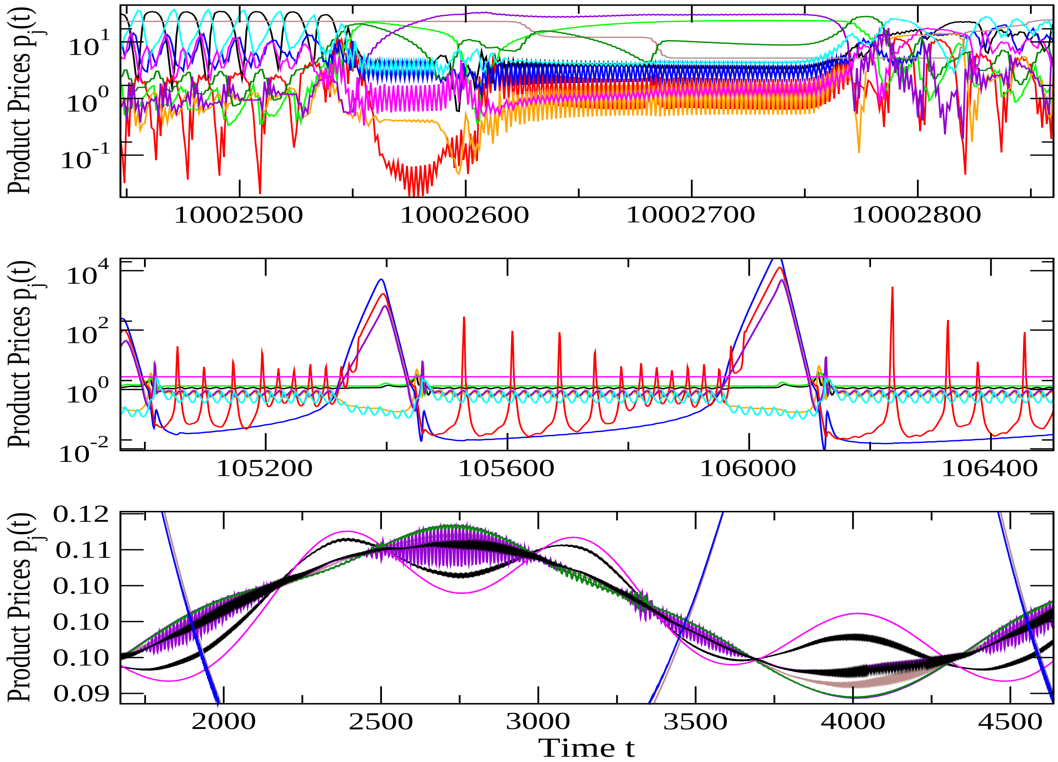

Fig. 14. Examples of time series from random networks with 15 processes and 10 products.

$$
\begin{aligned}
p_C(t) &= \frac{S_{1C}(t) + Min(S_{2B}(t), S_{2D}(t)) + S_3C(t) - Min(S_{3C}(t), S_{3D}(t))}{1/2F_1(t) + 1/2F_3(t)} \\
&= \frac{S_{1C}(t) + S_{2D}(t) + S_3C(t) - S_{3D}(t)}{1/2F_1(t) + 1/2F_3(t)} \sim \frac{S_3C(t)}{1/2F_3(t)} \qquad (32)
\end{aligned}
$$

so that the high frequency oscillations of the process 1 variable are "swamped" by the much larger magnitude of process 3 variables which do not have high the frequency component, and $p_C(t)$ does not show the high frequency oscillation.

This combination of bifurcations between different frequency states and the filtering of oscillations by threshold type prices, when some processes have much larger magnitudes than others, means time series from random economic networks show novel multiple timescale complex switching dynamics.

3.4 Dynamics of Random Networks

We do not go into the details of the dynamics of random networks of processes except to show some examples of such time series. In fact the strong attractor switching described above now occurs on multiple timescales and we find some processes controlling bifurcations in other processes in a very complex hierarchical way. The resultant dynamics, as seen in Fig.14 is extremely complex and will be addressed in future work.

4 Discussion

We have described a model of an economic production network and illustrated its very complex surprising behaviour with reference to some simple examples.

These examples have all been systems in a fixed environment. In reality we expect processes to respond and adapt to each other as described in Fig.2(b). When this is the case, for example a processes demanding catalyst will influence the catalyst supply process to increase its production, by increasing D_O^{ext} in the catalyst supply process production R^{equil}, Eq.16. To some extent this would seem to alleviate the problem, however in forthcoming work we show that this infact leads to even greater complexity, with complex multiple timescale threshold dynamics being typical, [Ponzi et al(2003.5)].

This model has provided a possible reason for the existence of business cycles, price bubbles and crashes, as basically due to the way processes can affect their own value by affecting product prices through demands for the products which they utilise. The sudden unpredictable threshold dynamics thereby created may be endemic in economics. Indeed we strongly suspect that the same unpredicatability will be realized no matter how 'intelligently' the process allocates its demands $\sigma_{ij}(t)$. If we consider that the single process could represent a highly idealized single country, turning imported product into exported product, using the catalyst of labour force, our model can offer an explanation for the ubiquitous cycles of macroeconomic growth and recession, unemployment and excess supply.

We also note that we have considered a highly idealized version of this system where we have not considered waste production, wear and tear, or heat production in any way. In reality catalysts are never perfectly conserved by the process but gradually get broken, and all production processes must also produce waste. It is easy to incorporate waste in or model in various ways, the simplest being to replace the output coefficients b_{ij} in Eq.7 by βb_{ij} where $0 < \beta < 1$. When we do this we find that for example, the price equilisation relationships Eq.23 no longer hold exactly, and we expect output prices will be larger than input prices to account for waste. Indeed the extra cost of the catalyst decay rate will be transferred to increased output price, so output price will depend on catalyst supply and demand rates. Waste production is also seen to reduce some of the more dramatic price bubbles and crashes shown by our model. This work is in progress.

Random network simulations of this model also show a variety of phase transitions as the network structure, and other parameters are varied. In particular we find a phase of complete economic collapse when there are too few processes to support the amount of products in the economy [A.Ponzi (2003)]. We hope to use this model to address questions such as what types of network structure are stable?, under what conditions can subsitute products coexist?, when can multiple processes producing the same output, or consuming the same input, coexist?, how many products can a given amount of processes support? Other questions we can address concern the optimal organization of factory supply lines, when they are also subject to an environment of fluctuating supply and demand which is in part created by the feedback of the supply chain dynamics onto the environment. Such topics have recently been discussed in a workshop, "Networks of Interacting Machines: Industrial Production Systems and Biological Cells" [A.Ponzi (2003.12)] with possible applications to organization of Intel Corporation factories and the understanding of the organization of biological cell structure. This work is in progress.

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