Discussion Paper No. 002

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November, 2003

21COE
Interfaces for Advanced Economic Analysis
Kyoto University

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The date of receipt and acceptance will be inserted by the editor

Abstract We investigate a consumption-investment problem in the setting of corporate finance considering a single agent disposing production possibilities. He can invest funds into both manufacturing and financial assets diversifying the income. The agent, endowed with an initial fund as well as initial production assets, strives to maximize the total expected utility from consumption over the finite time horizon. We establish for this problem a separation theorem. Namely, it can be solved by a two-stage procedure. The first stage is an independent optimization problem for the manufacturing arm and the second one is a standard Merton consumption-investment (portfolio selection) problem. The input parameter of the latter, namely, the initial budget, is determined by the optimal value of the manufacturing problem. Our analysis uses the Bismut stochastic maximum principle. In the case of deterministic coefficients and absence of random fluctuations the first problem is a classical deterministic problem which can be analyzed by the classical Pontriagin maximum principle. In particular examples we obtain closed form solutions.

Key words Consumption–investment problem, portfolio, production, stochastic equation, martingale, backward stochastic differential equation, Bismut stochastic maximum principle, Pontriagin maximum principle

Mathematics Subject Classification (2000): 60G44

JEL Classification Numbers: G13, G11

⋆ This research was done during the stay of the author at Daiwa Chair of Graduate School of Economics, Kyoto University.
⋆⋆ This author is supported in part by a Grant-in-Aid for the 21st Century COE research.

1 Introduction

We consider here a consumption–investment decision problem for a single "small" economic agent which can be viewed as a firm having production and financial arms. The initial endowment is in both assets. The problem is to maximize the total expected utility of the consumption rate over a finite time interval $[0, T]$ investing into the production as well as in the financial assets. It is assumed that the agent has an access to a frictionless security market with $d + 1$ assets, one of which is riskless and the others are risky. The market model is fairly standard: it is of the same type as in Karatzas et al. [9], see also Cox and Huang [4] and the expository paper [8]. Allocating the resources, the agent may invest funds into m production assets. This type of assets has features different from that of financial assets in the following two points. The investments into the manufacturing arm are non-reversible. The profit flow from the production at time t is $R(t, K_t)$ where $K_t = (K_t^1, ..., K_t^m)$ is the capital accumulation. The latter subjects random depreciations and, eventually, fluctuations due to external factors. The production assets cannot be cashed back before the terminal date T when the production arm can be sold at the price $Q(K_T)$. A similar problem was considered by Hirayama and Kijima in [7].

The agent in this model may be an owner of a small firm that produced some production goods. The consumption in this case can be interpreted as the dividend flow from the firm. The owner does not want to sell the business, since the ownership for him is very important (this is rather typical, especially, in such country as Japan). The role of the owner is to maximize the total utility from dividend. To do so, the owner may want to invest the limited fund in the production assets as much as possible to earn higher profits. But, since there is a financial market, he may also allocate a part of his wealth in securities. The problem for the owner is to decide portfolio strategy, dividend strategy, and production strategy so as to maximize the objective.

As we mentioned already, without the production arm, our model is reduced to the mainstream continuous-time portfolio optimization problem started in the famous papers by Merton [14], [15] and developed further in numerous publications (see, e.g., [4], [8], [9], [10], [16] and references therein). Production models were considered in [13] but without financial investments while the equilibrium approach to production economies was discussed in [18]. In real economies, firms invest their surplus funds in financial assets. It seems of interest to study optimal strategies in this more general context.

In our presentation we try to avoid technicalities. That is why we work with the easiest treated hypotheses, preferring, e.g., the boundedness assumption on coefficients to that of integrability. Our main message is that for the linear model with concave utility and production functions the problem can be split into two separate stages. First, the optimal production investment process $I^o = (I_t^o)$ can be found independently of the other counterparts of the optimal control as the optimal solution of a certain auxiliary control problem. Finding I^o, we have to solve, as the second stage, a classical portfolio problem which, as well-known, consists itself of two separate parts: a search for the optimal consumption and a search for the optimal investment (that is why we can say also that the whole problem has three stages).

This separation principle is our main contribution. We investigate the existence of the optimal solution for the auxiliary problem and derive necessary and sufficient conditions of optimality in the form of the Bismut maximum principle.

We investigate in more details a particular case of the model where the production block is not directly influenced by random perturbations. In this case the first stage is a deterministic control problem which can be analyzed on the basis of the Pontryagin maximum principle. We give examples where the optimal production policy is of the bang–bang type.

2 Model Description

We shall work in the standard probabilistic framework assuming that the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$ is fixed and the filtration is spanned by a d-dimensional Wiener process W.

First, we describe the production arm of the firm. It disposes m assets and if $K \in \mathbf{R}^m$ is a vector of values of these assets, the rate of the profit flow at time t is $R(t, K)$. The production asset i is depreciated with the rate λ^i which is, in general, a non-negative bounded predictable process. Its value also may fluctuate due to external factors. The capital accumulation evolves according to the stochastic differential equations

$$dK^i_t = (I^i_t - \lambda^i_t K^i_t)dt + K^i_t dL^i_t, \qquad K^i_0 = k^i, \tag{1}$$

where L is a martingale with

$$dL^i_t = \sum_{j=1}^{d} \sigma^{ij}_t dW^j_t, \qquad i \le m,$$

for some bounded predictable matrix-valued process σ.

The investments are assumed to be irreversible, i.e. the capital accumulation may increase only by depreciation and by random fluctuations (if $\sigma = 0$ the later are not taken into account). The production strategy I is a predictable process with values in a bounded compact convex subset Γ of $\mathbf{R}^m_+$. We shall assume that there exists a suitably integrable random variable ζ such that for any capital accumulation process K

$$\int_0^T |R(s, K_s)| ds \le \zeta. \tag{2}$$

As we shall see further, the required property is $\zeta \in L^1(\tilde{P})$ where $\tilde{P}$ is the equivalent martingale measure.

The production assets cannot be sold before T, but they can be liquidated at the price $Q(K_T)$ at the terminal date. It is natural to assume that in the variable K the functions R and Q are concave and increasing (componentwise).

The agent also has an access to a frictionless financial market of the Black–Scholes type with $d+1$ securities. One of them is non-risky ("bond" or "bank account") and has the price evolving as

$$\frac{dP_t^0}{P_t^0} = r_t dt, \qquad P_0^0 = p^0 = 1. \tag{3}$$

For simplicity, mainly, notational, we suppose from the very beginning that $r=0$, i.e. bond is the numéraire and all investments are measured in its units.

The prices of remaining assets, (risky) stocks, are modelled by the stochastic equations

$$\frac{dP_t^i}{P_t^i} = b_t^i dt + dM_t^i, \qquad P_0^i = p^i, \tag{4}$$

where M is a square integrable martingale generating our basic filtration $\mathbf{F}$ (of the Wiener process W). We assume more specifically that

$$dM_t^i = \sum_{j=1}^{d} \Sigma_t^{ij} dW_t^j, \qquad i \le d.$$

The vector of instantaneous rate of returns b and the (non-degenerate) volatility matrix Σ and its inverse Σ^{-1} are assumed to be bounded predictable processes.

The agent's portfolio at date t contains n_t^i units of the asset i. His holdings in risky assets of the financial market $\pi_t^i = n_t^i P_t^i$, $1 \le i \le d$, are predictable processes such that

$$\int_0^T |\pi_t|^2 dt < \infty.$$

The agent consumption intensity is a predictable non-negative process $c=(c_t)$ with

$$\int_0^T c_t dt < \infty.$$

The triplet of the investment processes and consumption $u=(\pi, I, c)$ is the control strategy. The optimization problem can be formulated as:

$$E\int_0^T e^{-\beta t} U(c_t) dt \to \max, \tag{5}$$

with the controlled dynamics of the total fund given by the following stochastic differential equation where $\mathbf{1} := (1, ..., 1)$:

$$dX_t = (R(t, K_t) - \mathbf{1} I_t - c_t)dt + \pi_t(b_t dt + dM_t), \qquad X_0 = x. \tag{6}$$

To avoid technicalities, we suppose that the utility function $U : \mathbf{R}_+ \to \mathbf{R}_+$ in (5) is a concave increasing function with $U(0)=0$, $U'(0)=\infty$ and $U'(\infty)=0$ (note that U is differentiable everywhere except at most a countable number of points).

In addition to the constraints indicated above we impose a constraint on the controls which prevents a "bankruptcy" before the date T. Namely, we shall consider as admissible only the controls u such that

$$V_t := X_t + \tilde{E}\left[\int_t^T R(s,K_s)ds + Q(K_T)|\mathcal{F}_t\right] \geq 0, \qquad \forall\, t \leq T. \tag{7}$$

The symbol $\tilde{E}$ indicates that the expectation is taken with respect to the (unique) martingale measure $\tilde{P}$. The corresponding term can be interpreted as the market evaluation of the manufacturing arm of the company. This makes plausible the assumption that the agent may borrow funds until this level.

The set of admissible strategies depends on the initial endowment $y := (x,k)$. It will be denoted by $\mathcal{A}(y)$.

We shall assume that $\mathcal{A}(y) \neq \emptyset$, i.e. at least one admissible strategy u does exist. Obviously, this is always the case when R and Q are non-negative, since $u = (0,0,0)$ belongs to $\mathcal{A}(y)$.

Recall that $\tilde{P} = Z_T P$ where

$$Z_t = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right\},$$

with $\theta_s := -\Sigma_s^{-1} b_s$. Under $\tilde{P}$

$$\tilde{W}_t := W_t - \int_0^t \theta_s ds$$

is a Wiener process.

With this the dynamics of the phase variable (6) can be rewritten as follows

$$X_t = x + \int_0^t (R(s,K_s) - \mathbf{1}I_s - c_s)ds + \int_0^t \pi_s d\tilde{M}_s, \tag{8}$$

where both $\tilde{M}$ and the stochastic integral in the above formula are square integrable martingales with respect to $\tilde{P}$. Thus,

$$V_t = x + \tilde{E}\left[\int_0^T R(s,K_s)ds + Q(K_T)|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds + \int_0^t \pi_s d\tilde{M}_s.$$

The definition of admissibility implies, in particular, that $\tilde{E}V_T \geq 0$. We write this inequality in the equivalent form

$$\tilde{E}\int_0^T c_s ds \leq x - H(I) \tag{9}$$

where

$$H(I) := \tilde{E}\left[\int_0^T (\mathbf{1}I_s - R(s,K_s))ds - Q(K_T)\right]. \tag{10}$$

Let us denote by $\mathcal{C}(y)$ the set of pairs of production and investment processes (I, c) for which (9) holds.

The next lemma is established in the same way as in the classical consumption–investment model, see, e.g., the textbook [11].

Lemma 1 *For any given $(I, c) \in \mathcal{C}(y)$ there exists a portfolio process π such that $(\pi, I, c) \in \mathcal{A}(y)$.*

Proof. Let $(I, c) \in \mathcal{C}(y)$. Consider the non-negative process V with

$$V_t := \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s)ds|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds$$
$$+x - \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s - R(s, K_s))ds - Q(K_T)\right].$$

It can be written in the form

$$V_t = x + \tilde{E}\left[\int_0^T R(s, K_s)ds + Q(K_T)|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds + M_t^V - M_0^V.$$

where

$$M_t^V := \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s - R(s, K_s))ds - Q(K_T)|\mathcal{F}_t\right].$$

By the martingale representation theorem

$$M_t^V - M_0^V = \int_0^t \pi_s d\tilde{M}_s$$

and we infer easily from (7) and (8) that the triplet $(\pi, I, c) \in \mathcal{A}(y)$.

The conclusion following from this lemma is very important: solving the original problem with a seemingly complicated "pointwise" constraint (7) is reduced to the solving of a much simpler problem with a single "traditional" inequality constraint given by a convex functional, with a consequent search for the corresponding investment strategy. Moreover, it is easily seen that the search for the optimal production and optimal consumption also can be done in a separate consecutive way. Indeed, since the utility function is increasing, for a given production strategy I with $H(I) \le x$ (such a strategy exists as there is an admissible strategy u), the corresponding maximal value of the functional is attended on a consumption strategy for which (9) holds with the equality. The maximal possible value will correspond to I^o on which $H(I)$ attains minimum. The existence of the optimal I^o as well as the solution of the consumption problems satisfying (9) follows from the Komlós theorem - we recall the arguments in the next section dealing with the optimal production strategy. Summarizing, we arrive to the following

Proposition 2 *In the optimal solution $(\pi^o, I^o, c^o) \in \mathcal{A}(y)$ of the consumption-investment problem with production possibilities the optimal investment I^o in manufacturing arm is the the minimizer for the problem with the functional (10) and the dynamics (1). The optimal consumption process $c^o \geq 0$ is the solution of the maximization problem (5) under the constraint (9). The optimal portfolio strategy π^o is the unique square-integrable predictable process satisfying the identity*

$$M_t^{V^o} = M_0^{V^o} + \int_0^t \pi_s^o d\tilde{M}_s$$

with

$$M_t^{V^o} := \tilde{E}\left[\int_0^T (\mathbf{1} I_s^o + c_s^o - R(s, K_s^o))ds - Q(K_T^o)|\mathcal{F}_t\right].$$

3 Optimal Production Investment

Let us consider separately the optimal control problem

$$H(I) := \tilde{E}\left[\int_0^T (\mathbf{1} I_s - R(s, K_s))ds - Q(K_T)\right] \to \min \tag{11}$$

over the convex set $\mathcal{I}$ of all Γ-valued predictable processes I and where K is given by (1). This problem belongs to the well-studied class of convex problems for which one can use duality methods.

Now standard (and fast) way to prove the existence in the convex optimal control problems is the reference to the Komlós theorem. The latter claims that for any L^1-bounded sequence of random variables ξ_n there exist a random variable $\xi \in L^1$ and a subsequence ξ_{n_k} converging to ξ a.s. in the Cesaro sense.

Let $H^o = \inf_{I \in \mathcal{I}} H(I)$ and let $H(I^n) \to H^o$ for some $I^n \in \mathcal{I}$. Due to the boundedness of Γ we can apply the Komlós theorem to I^n considering these processes as random variables on the space $(\Omega \times [0, T], \mathcal{P}, d\tilde{P}dt)$, where $\mathcal{P}$ is the predictable σ-algebra. Renumbering, we may assume without loss of generality that the original sequence converges $d\tilde{P}dt$-a.e. to some $I \in \mathcal{I}$ in Cesaro sense. This means simply that the controls $\bar{I}^n := n^{-1}\sum_{j=1}^n I^j$ converge (a.e.) to I^o which is, clearly, an element of $\mathcal{I}$. In virtue of the hypothesis (2) and continuity of R we obtain by the dominated convergence that $H(I^o) = \lim H(\bar{I}^n)$. Due to the convexity of H (following from the linearity of the equation (1) and concavity of R and Q) we have that

$$H(I^o) = \lim H(\bar{I}^n) \leq \lim n^{-1}\sum_{j=1}^n H(I^j) = H^o.$$

Thus, $H(I^o) = H^o$, i.e. I^o is the optimal control.

To understand the structure of the optimal control, we apply the Bismut stochastic maximum principle, see [2], [3]. To this end, isolating the $\tilde{P}$-martingale term and using the abbreviation $\mu_t := \lambda_t - \sigma_t\theta_t$, we rewrite the dynamics of manufacturing capital as

$$dK_t = (I_t - \operatorname{diag} K_t\mu_t)dt + \operatorname{diag} K_t\,\sigma_t d\tilde{W}_t, \qquad K_0 = k, \tag{12}$$

and introduce the Hamiltonian

$$\mathcal{H}(t,K,I,p,h) := \langle p, I - \operatorname{diag} K\,\mu_t\rangle + \langle h, \operatorname{diag} K\,\sigma_t\rangle + R(t,K) - \langle \mathbf{1}, I\rangle,$$

where $p \in \mathbf{R}^m$ and h is an $m \times d$-matrix. Exceptionally, we use here the notation $\langle .,.\rangle$ for scalar products following the traditional form easy to memorize which was suggested by Bismut. Note that the second term can be written as $\operatorname{tr} h(\operatorname{diag} K\,\sigma_t)^*$, where * denotes transpose and tr the trace.

The theorem claims that the pair (I^o, K^o) satisfying the equation

$$dK^o_t = (I^o_t - \operatorname{diag} K^o_t\mu_t)dt + \operatorname{diag} K^o_t\sigma_t d\tilde{W}_t, \qquad K^o_0 = k, \tag{13}$$

is optimal for the problem (11), (12) if there exist a continuous predictable processes p with square integrable sup norm and a process $h \in L^2(\Omega\times[0,T],\mathcal{P},d\tilde{P}dt)$ solving the m-dimensional backward stochastic differential equation (BSDE)

$$dp_t = -\nabla\mathcal{H}(t,K^o_t,I^o_t,p_t,h_t)dt + h_t d\tilde{W}_t, \qquad p_T = \nabla Q(K^o_T), \tag{14}$$

where ∇ is the gradient in the variable K, specifically,

$$dp_t = (\operatorname{diag}\mu_t\;p_t - \nabla R(t,K^o_t) - \widehat{h}_t)dt + h_t d\tilde{W}_t, \qquad p_T = \nabla Q(K^o_T), \tag{15}$$

where $\widehat{h}^i_t = \sum_j h^{ij}_t\sigma^{ij}_t$ and the following relation holds:

$$\mathcal{H}(t,K^o_t,I^o_t,p_t,h_t) = \max_{I\in\Gamma}\mathcal{H}(t,K^o_t,I,p_t,h_t) \qquad d\tilde{P}dt\text{-a.e.} \tag{16}$$

For brevity we shall call any quadruplet of processes I^o, K^o p, and h satisfying the above relations and the integrability assumption a *Bismut quadruplet*. The convexity of the considered minimization problem implies that the Bismut quadruplet is unique and I^o is the optimal control.

Let I be an arbitrary Γ-valued predictable process. Using (13) and (15) we get by the Ito formula that

$$\begin{aligned} d(p_tK_t) &= (p_t\operatorname{diag}\mu_t\,K_t - \nabla R(t,K^o_t)K_t - \operatorname{tr} h(\operatorname{diag} K\,\sigma_t)^*)dt \\ &\quad + p_t(I_t - \operatorname{diag} K_t\mu_t)dt + \operatorname{tr} h(\operatorname{diag} K\,\sigma_t)^*dt + dN_t \\ &= (p_t(I_t - \nabla R(t,K^o_t)K_t)dt + dN_t \end{aligned}$$

where N is a square integrable martingale with respect to $\tilde{P}$.

Writing this in the integral form and observing that the expectation of stochastic integral vanishes we arrive to the formula

$$\tilde{E}\int_0^T p_tI_tdt = \tilde{E}\nabla Q(K^o_T)K_T - p_0k + \tilde{E}\int_0^T \nabla R(t,K^o_t)K_tdt.$$

This formula holds, in particular, for I^o and K^o. Taking the difference and using the concavity of R and Q, we obtain easily that

$$\tilde{E}\int_0^T p_t(I_t^o - I_t)dt \le \tilde{E}\int_0^T (R(t,K_t^o) - R(t,K_t))dt + \tilde{E}(Q(K_T^o) - Q(K_t)). \tag{17}$$

But the maximum principle (16) implies

$$\int_0^T \mathbf{1}(I_t^o - I_t)dt \le \int_0^T p_t(I_t^o - I_t)dt \qquad \tilde{P}\text{-a.s.} \tag{18}$$

and we deduce from these two inequalities that $H(I^o) \le H(I)$.

Due to the simplicity of our problem we can see easily that the stochastic maximum principle is the necessary condition: the optimal pair is the component of a Bismut quadruplet. Indeed, starting from the optimal pair (I^o, K^o) we can define p and h satisfying (15). The optimality of (I^o, K^o) implies that in (17) and (18) we have equalities. But (18) is equivalent to (16).

Summarizing, we have the following.

Proposition 3 *A pair (I^o, K^o) satisfying (13) is an optimal solution of the problem (10), (12) if and only if it can be complimented to a Bismut quadruplet.*

In the case where $\sigma = 0$ and, therefore, h appears only in the diffusion term, the linear backward equation can be "solved" easily. Indeed, the m-dimensional random variable

$$\xi := \int_0^T e_s^{-\lambda}\nabla R(s,K_s^o)ds + e_T^{-\lambda}\nabla Q(K_T^o)$$

with

$$e_t^\lambda := \operatorname{diag}\left\{e^{\int_0^t \lambda_s^1 ds}, ..., e^{\int_0^t \lambda_s^m ds}\right\}$$

is a square integrable functional of the Wiener process. By the martingale representation theorem

$$\tilde{E}(\xi|\mathcal{F}_t) = \tilde{E}\xi + \int_0^t \phi_s dM_s$$

for some matrix-valued process $\phi \in L^2(\Omega \times [0,T], \mathcal{P}, d\tilde{P}dt)$ of an appropriate dimension. It is easy to see that $h_t := e_t^\lambda \phi_t$ and

$$p_t := e_t^\lambda \tilde{E}\xi - e_t^\lambda \int_0^t e_s^{-\lambda}\nabla R(s,K_s^o)ds + e_t^\lambda \int_0^t \phi_s dM_s$$

is the solution of the backward stochastic equation (15).

In the case $d = 1$ we can get an "explicit" solution of the BSDE for arbitrary σ by making at first the equivalent change of the probability measure, removing the term $\widehat{h}$ from the drift (under this measure the process with $d\tilde{W}_t' := d\tilde{W}_t + \sigma_t dt$ Wiener). In general case we use just a reference to an existence theorem for the solution of a linear BSDE. An appropriate result can be found, e.g., in [5].

4 Special Cases

4.1 Deterministic Dynamics: Examples.

The separation principle has an important consequence for the case of the model where the values of the production assets may only depreciate (i.e. $\sigma = 0$) and the parameters λ^i are deterministic. The problem becomes deterministic:

$$H(K) := \int_0^T (\mathbf{1} I_t - R(t, K_t))dt - Q(K_T) \to \min, \tag{19}$$

$$\dot{K}^i_t = I^i_t - \lambda^i_t K^i_t, \qquad K^i_0 = k^i, \tag{20}$$

where $I = (I_t)$ is a Borel function taking values in $\Gamma \subset \mathbf{R}^m_+$.

The necessary and sufficient condition of optimality is the classical Pontriagin maximum principle. More specifically, a pair (I^o, K^o) is optimal for the problem (19), (20) if and only if it is a part of the "Pontryagin triplet" (I^o, K^o, p) satisfying the following relations:

$$\dot{K}^o_t = I^o_t - \operatorname{diag} \lambda_t K^o_t, \qquad K^o_0 = k, \tag{21}$$

$$\dot{p}_t = p_t \operatorname{diag} \lambda_t - \nabla R(t, K^o_t), \qquad p_T = \nabla Q(K^o_T), \tag{22}$$

$$(p_t - \mathbf{1}) I^o_t = \max_{I \in \Gamma} (p_t - \mathbf{1}) I_t \quad a.e. \tag{23}$$

Due to the number of parameters involved, the complete analysis of this system seems to be rather complicated. We restrict ourselves to the scalar problem with constant coefficients and $\Gamma = [0, a]$ and provide several examples where the solution can be obtained explicitly. For $m = 1$ we have:

$$\dot{K}^o_t = I^o_t - \lambda K^o_t, \qquad K^o_0 = k, \tag{24}$$

$$\dot{p}_t = \lambda p_t - R'(K^o_t), \qquad p_T = Q'(K^o_T), \tag{25}$$

$$(p_t - 1) I^o_t = \max_{I \in \Gamma} (p_t - 1) I \quad a.e. \tag{26}$$

Let us investigate the case where $Q = \text{const}$ (such a situation may arise in practice) and $R(K) = (\kappa/\gamma) K^\gamma$, $\kappa > 0$, $\gamma \in]0, 1[$. Due to the continuity, near the right extremity T of the time interval the dual variable p is close to the value $p_T = 0$; more precisely, in virtue of the equation (25), it decreases to zero. Let $T_1 := \sup\{t \geq 0 : \ p_t \geq 1\}$ (with the convention that $T_1 = 0$ if the set is empty). The maximum relation ensures that $I^o_t = 0$ on $]T_1, T]$. If $T_1 = 0$, the phase trajectory is the decreasing exponential $K^o_t = k e^{-\lambda t}$ while the trajectory of the dual variable is

$$p_t = e^{\lambda t} \int_t^T e^{-\lambda s} R'(K^o_s) ds = k^{\gamma - 1} \frac{\kappa}{\lambda \gamma} e^{\lambda t} (e^{-\lambda \gamma t} - e^{-\lambda \gamma T}).$$

To be compatible with the maximum principle the right-hand side should be less or equal to unity on the whole interval $[0, T]$ and this requirement is met when the initial endowment $k \geq k^c$ where the threshold is given by

$$k^c = \sup_{t \leq T} \left[\frac{\kappa}{\lambda\gamma} e^{\lambda t} (e^{-\lambda\gamma t} - e^{-\lambda\gamma T}) \right]^{\frac{1}{1-\gamma}}.$$

Thus, for large k the control $I_t^o = 0$. We shall have, for large initial endowments in production assets, the similar structure of the optimal control also for the model where $Q'(K) \to 0$ as $K \to \infty$.

Qualitatively, this result means that in the case of small marginal liquidation value the investor having high level of initial manufacturing facilities is not motivated in their further development.

The situation seems to be rather different for $k < k^c$. Then necessarily I^o is not equal to zero on a certain non-null subset of $[0, T_1]$. Let us show that for some range of parameters, $I_t^o = aI_{[0,T_1]}$.

So, suppose that on $[0, T_1]$ the control $I_t^o = a$ and, therefore, on this interval the state dynamics is given by the formula

$$K_t^o = ke^{-\lambda t} + \frac{a}{\lambda}(1 - e^{-\lambda t}) = \frac{a}{\lambda} + \left(k - \frac{a}{\lambda} \right) e^{-\lambda t}. \tag{27}$$

First, we consider the simplest particular case where $k = a/\lambda$. Then $K_t^o = k$ on $[0, T_1[$ (the maximal level of investments keeps the production capacity constant) and, according to (25), $\dot{p}_{T_1} = \lambda - \kappa k^{\gamma-1}$. For $t \in [T_1, T]$ we have the formula $K_t^o = ke^{\lambda T_1} e^{-\lambda t}$ and, hence, on this interval

$$p_t = k^{\gamma-1} e^{\lambda(\gamma-1)T_1} \frac{\kappa}{\lambda\gamma} e^{\lambda t} (e^{-\lambda\gamma t} - e^{-\lambda\gamma T}).$$

Note that the point $T_1 \in]0, T[$ can be defined from the equation $p_{T_1} = 1$ which solution does exist for $k < k^c$. On the interval $[0, T_1]$ the function p solving the differential equation

$$\dot{p}_t = \lambda p_t - \kappa k^{\gamma-1}, \qquad p_{T_1} = 1,$$

and hence given by the formula

$$p_t = \frac{\kappa}{\lambda} k^{\gamma-1} + \left(1 - \frac{\kappa}{\lambda} k^{\gamma-1} \right) e^{-\lambda(T_1 - t)}$$

should be larger or equal to unity. If also $k < (\kappa/\lambda)^{\frac{1}{1-\gamma}}$, the value of derivative $\dot{p}_{T_1} < 0$. Taking into account that the trajectory cannot cross the unit level upwards with negative value of derivative (always equal to $\lambda - \kappa k^{\gamma-1}$), we conclude that the control $aI_{[0,T_1]}$ is optimal for such values of the initial endowment k.

If $k > a/\lambda$, the trajectory supposed to be optimal decreases on $[0, T_1]$ from its initial value k. For $k < (\lambda/\kappa)^{\frac{1}{1-\gamma}}$, we have $\dot{p}_{T_1} < 0$, i.e. the dual variable cross the unit level at T_1 and cannot do this before.

If $k < a/\lambda$, the candidate for the optimal trajectory on $[0, T_1]$ increases from k to a certain value which is less than a/λ. At least, in the case of the small ratio

a/λ (i.e., when $\lambda < \kappa(a/\lambda)^{\gamma-1}$), we can conclude again that $p_t > 1$ on $[0,T_1[$ and, therefore, $I^o_t = aI_{[0,T_1]}$ is the optimal control.

In short, for initial endowments k less than a certain critical value k_c (in some case, with appropriate restrictions on other parameters), the optimal strategy is of the bang-bang form and requires at the beginning of the planning interval intensive investments in the production assets.

However, in the range $]k_c, k^c[$ the structure of the optimal control may be more involved and even not of the bang-bang type.

4.2 Deterministic Dynamics: Turnpike Behavior

To investigate the general structure of the optimal control in the problem (19), (20), we exclude the control variable from the functional using the expressions $I^i_t = \dot K^i_t + \lambda^i_t K^i_t$ given by (20). After simple transformations we arrive to the problem with the functional depending only of the phase variable:

$$\int_0^T \Phi(t,K_t)dt + S(K_T) \to \min, \tag{28}$$

$$\dot K^i_t = I^i_t - \lambda^i_t K^i_t, \qquad K^i_0 = k^i, \tag{29}$$

where the functions $\Phi(t,K) := \lambda_t K - R(t,K)$ and $S(K) := \mathbf{I}K - Q(K) - \mathbf{I}k$ are convex and increasing in K.

It is well-known that, under minor assumptions, the optimal trajectory in models of such type exhibits, on a large time interval, a turnpike behavior: it coincides, except initial and final periods, with the function $\widehat K$ where $\widehat K_t$ is the minimizer of the function $\Phi(t,.)$, i.e. the root of the equation $\nabla\Phi(t,K) = 0$.

To be specific, we consider again the one-dimensional time-homogeneous model assuming also that $k < a/\lambda$, $\Phi'(a/\lambda) > 0$, $\Phi'(0) = -\infty$. Then any trajectory K evolves in the interval $[0,a/\lambda]$; it increases if $I = a$ and decreases if $I = 0$.

Now the dual variable $\psi = p - 1$ solves the equation

$$\dot\psi_t = \lambda\psi_t + \Phi(K^o_t), \qquad \psi_T = -S'(K^o_T). \tag{30}$$

and the maximum principle says that $I^o_t = 0$ if $\psi_t < 0$, and $I^o_t = a$ if $\psi_t > 0$. It is convenient to introduce an auxiliary function $q_t := e^{-\lambda t}\psi_t$ having the same sign as ψ_t; its derivative $\dot q_t = e^{-\lambda t}\Phi'(K^o_t)$.

Let $t_1 := \inf\{t :\ q_t = 0\}$, $t_2 := \sup\{t :\ q_t = 0\}$. Notice that if $[t_1,t_2]$ is not a singleton, then on this interval $q = 0$. Indeed, suppose that there is a subinterval $]t',t''[$ where $q < 0$ but $q_{t'} = q_{t''} = 0$. Since on this subinterval $I^o = 0$, the trajectory K^o is decreasing, $\Phi'(K^o)$ is also decreasing and so is $\dot q$. This is impossible and, therefore, q cannot deviate from zero downwards. Similarly, if $q > 0$ on $]t',t''[$ and q vanishes at the extremities, then on this interval $I^o = a$, the trajectory K^o increases as well as $\Phi'(K^o)$. Thus,

$$\dot\psi_{t'} = \Phi'(K^o_{t'}) < \Phi'(K^o_{t''}) = \dot\psi_{t''}$$

in contradiction with the inequalities $\dot\psi_{t'} \ge 0$, $\dot\psi_{t''} \le 0$.

The equation (30) necessitates that $\Phi'(K^o) = 0$ on $[t_1, t_2]$, i.e. $K^o = \widehat{K}$ where $\widehat{K}$ is the minimizer of Φ; the optimal control is $I^o = \widehat{K}/\lambda$. The left extremity coincides with zero if and only if $k = \widehat{K}$. If $t_1 > 0$, there are two possible cases: 1) on $[0, t_1[$ the dual variable ψ is strictly negative, $I^o = 0$ and the trajectory K^o decreases from k to the value $\widehat{K}$; 2) on $[0, t_1[$ the dual variable ψ is strictly positive, $I^o = a$ and the trajectory K^o increases from k to the value $\widehat{K}$. In both cases the interval $[0, t_1]$ does not depend on the terminal part of the functional and $t_1 < T$ for sufficiently large T.

The case $t_2 = T$ is exceptional. This means that $0 = \psi_T = -S'(\widehat{K})$, i.e., $\widehat{K}$ minimizes also the function S. Otherwise, the interval $[t_2, T]$ is not a singleton. The optimal control on this interval depends on the sign of $S'(\widehat{K})$. Suppose, e.g., that $S'(\widehat{K}) > 0$. Let $I^o = 0$. Then ψ is strictly negative, the trajectory K^o decreases from the value $\widehat{K}$, $\Phi(K^o) < 0$ and, therefore $\dot{\psi} = \lambda\psi + \Phi(K^o) < 0$, i.e., the trajectory ψ decreases from zero. Since $-S'$ is a decreasing function, the transversality condition $\psi_T = -S'(K^o_T)$ will be met for a certain (uniquely defined) value of t_2 (of course, the time horizon should be large enough).

The above arguments show that, for a long time interval, the optimal investments in the manufacturing consist in keeping the production on a specific "turnpike" level which depends only of the technology used and not of the initial capital and the liquidation value. This level should be attained in the fastest way at the beginning of the planning period. At the end of the period, the investment policy is to leave the turnpike quickly to profit from the selling of the manufacturing arm.

4.3 Remark on the HJB equation

The case where the fluctuations of the price of production assets are assumed (i.e. σ is not zero) can be studied by methods of dynamic programming. The problem of interest can be imbedded in the family of stochastic control problems parameterized by initial date t and the initial endowment x (we prefer x to k here for notational convenience). The HJB equation is as follows:

$$V_t + \inf_{I\in[0,a]} \left[\frac{1}{2}\sigma^2 x^2 V_{xx} + (I - \mu x)V_x + (I - R(x))\right] = 0$$

with the terminal condition $V(T, x) = -Q(x)$. The number H^o we are interested in is $V(0, k)$. The above equation can be rewritten in the form

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} - \mu x V_x + a I_{\{V_x \le -1\}} - R(x) = 0.$$

One can prove that the Bellman function V of the problem is a viscosity solutions of this equation which is unique in an appropriate class but a detailed discussion is beyond the scope of the present paper.

4.4 Piecewise-linear utility function

As we just see, in some cases the production problem may admit an explicit solution otherwise the value H^o can be found numerically. An attractive feature of the considered setting is that the investing problem is well-studied and also admits cases with explicit solutions. The most famous one is the problem with $U(c) = \rho/c^\rho$ found by Merton.

We discuss here an example where the utility function is linear up to a saturation point, i.e.

$$U(c) = cI_{\{c \le C\}} + CI_{\{c > C\}}.$$

Thus, the optimal control problem is read now:

$$J(c) := E \int_0^T e^{-\beta t} U(c_t)dt \to \max$$

over all non-negative predictable processes c such that

$$E \int_0^T Z_t c_t dt \le x - H(I^o).$$

Clearly, in our search for the optimum we can consider the subset of controls for which the constraint is satisfied with an equality.

The solution can be found easily using the Lagrange multiplier method removing the above constraint. Arguing formally, we write the unconstrained problem

$$E \int_0^T [e^{-\beta t} U(c_t) - \theta Z_t c_t]dt \to \max$$

where the multiplier $\theta \ge 0$. Its solution is any non-negative predictable process $c = (c_t)$ maximizing pointwise the integrand. Of course, the solution depends of the unknown Lagrange multiplier θ. Let

$$c_t^*(\theta) := CI_{\{\theta Z_t > e^{-\beta t}\}}.$$

Define on $\mathbf{R}_+$ the function

$$f(\theta) := E \int_0^T Z_t c_t^*(\theta)dt = C \int_0^T \tilde{P}(e^{\beta t} Z_t < 1/\theta)dt$$

which is continuous and decreasing from $f(0) = CT$ to $f(\infty) = 0$.

Let us show that the optimal consumption process is $c^o := c^*(\theta^*)$ where θ^* is defined as the solution of the equation $f(\theta^*) = x - H(I^o)$ and this solution we assume existing (otherwise the problem is trivial with the optimal solution $c_t^o = C$). Indeed, let $c = (c_t)$ be an arbitrary consumption process satisfying the constraint with the equality. Then

$$J(c^o) - J(c) = E \int_0^T [e^{-\beta t} U(c^o) - \theta^* Z_t c_t^o - e^{-\beta t} U(c_t) + \theta^* Z_t c_t]dt$$

and we get the result because the right-hand side is non-negative due to the choice of c^o as the maximizer of the unconstrained problem with the multiplier θ^*.

Acknowledgment. The authors are grateful to Andrei Dmitruk to whom they are indebted for the arguments on the turnpike behavior used in Subsection 4.2. His expertise in the Pontriagin maximum principle is greatly appreciated.

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